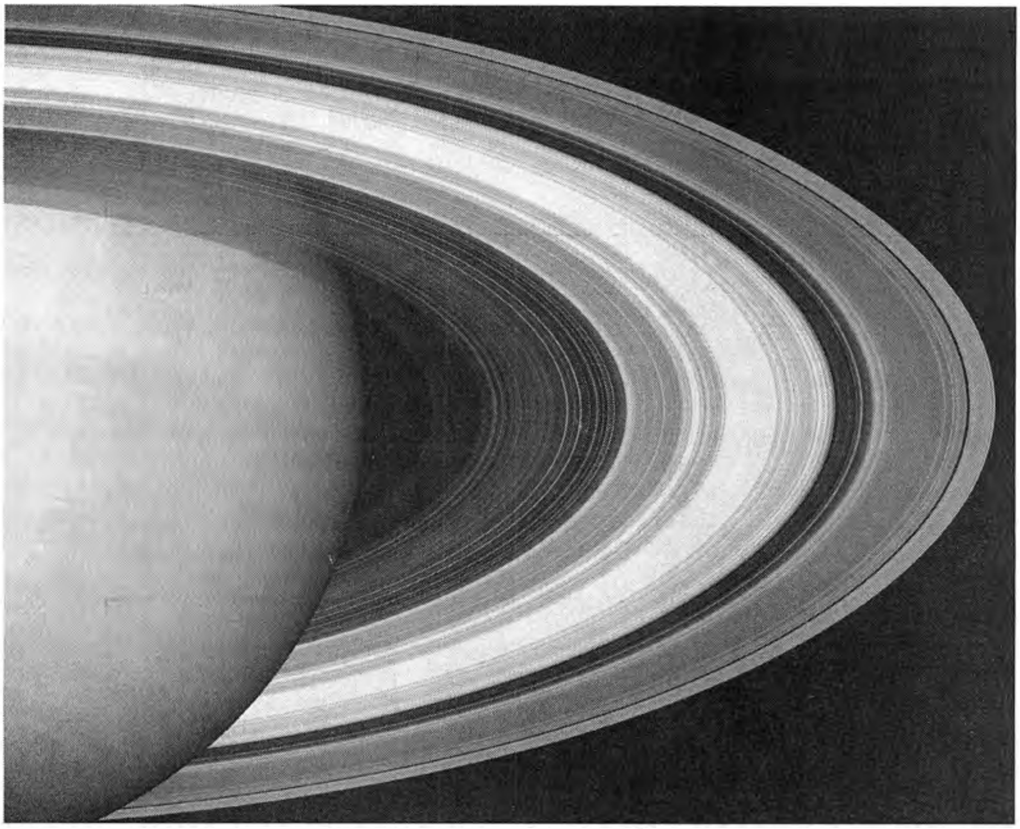
Archimedean Polyhedra And the Boundary: The Missing Link



NASA/JPL

by Hal Wm. Vaughan

There's more to the structure of space than meets the eye, as you'll see in this geometry adventure, which takes you to the limits of the universe.

A view of Saturn's rings. The study of the Platonic and Archimedean solids reveals that space has a structure, and that structure exposes a discoverable intention, which has created a boundary.

"Geometry is one and eternal, a reflection from the mind of God. That mankind shares in it is because man is an image of God."

—*Johannes Kepler*[1]

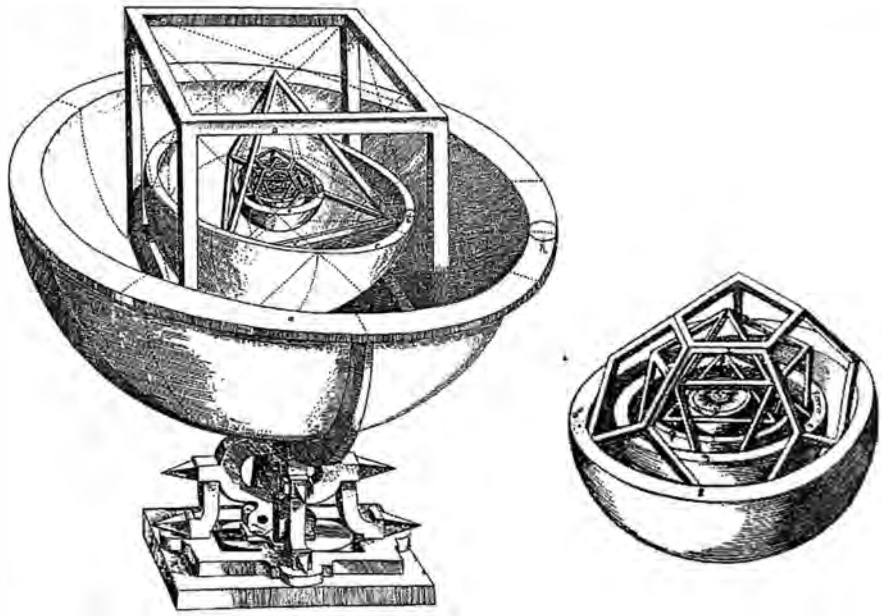
Keeping in mind the above invocation, we are going to develop, through a sometimes good-natured *analysis situs* of the Platonic and Archimedean polyhedra, an examination of the limits that constrain physical space. My contention is that the boundary demonstrated by the construction of the Platonic solids can not be fully apprehended without involving the Archimedean polyhedra in the investigation. This discourse is not meant to substitute for your working through the discoveries of Carl Gauss or Bernhard Riemann, but is meant to fill a conspicuous gap in existing pedagogy. The Archimedean polyhedra are largely, and for quite sensible reasons, an unexplored area of study, and on that basis, my subtitle is emblazoned above, for all to see, as "The Missing Link."

By the time we are done, we will have constructed the geometrical equivalent of an imaginary toolchest which will then be available for your use in later efforts. This chest has an array of tools, arranged in two different drawers. One set of tools is realized on the surfaces of three spheres and comes from a place somewhere "above" the spheres. The other set is ren-

Figure 1
KEPLER'S PLANETARY ORDERING

Johannes Kepler (1571-1630), who discovered the principle of gravitation during his studies of the movements of the planets in the Solar System, saw a coherence in the harmonious ordering of the planets in their orbits, and the harmonious ordering of the nested Platonic solids.

This is an engraving of Kepler's determination of the orbits of the planets, from his Mysterium Cosmographicum. *His ordering, beginning from the circumsphere defining the orbit of Mercury, are: octahedron, icosahedron, dodecahedron (of which the insphere is Earth and the circumsphere is Mars), tetrahedron, and cube.*



dered in two dimensions, even though it was developed from a three-dimensional lattice. I haven't invented any of these tools. Some of them have been known for decades, others for millennia, but the sets have never been assembled in this fashion before; nor, to my knowledge, has the insistence been presented that these tools, as sets, be used in the workshop of your mind.

Why Archimedean Polyhedra?

Study of the Platonic solids reveals that space is not just an endless checkerboard; it has a structure, and the structure exposes a discoverable intention, which has created a boundary.

There are five, and only five shapes that are convex polyhedra with regular, congruent faces whose edge-angles and vertices are equal: the Platonic solids (Figure 3). You can only make these five shapes within those constraints, and hence the limit. When you try to make more regular solids, say, by putting 6 triangles, or 4 squares together at a vertex, you don't get a solid at all; you can't do it, no matter how hard you try. The fact that your grand project of regular-polyhedron manufacture is brought to an abrupt halt after only five successes, says that there is more to the universe than meets the eye. Something in the make-up of everything you can see is different from *what* you see. That is the importance of the Platonic solids. They prove that we don't know what we are looking at.

The uniqueness of the Platonic solids proves that we are not living on a checkerboard at all; we are living in a goldfish bowl. The limits are real. Admittedly, most people spend their time looking at the rocks and bubbles in their bowl, or they choose to play checkers on the nonexistent checkerboard, and wonder how long it will be until feeding time.

I wanted to know what the shape of the fishbowl is. Just how do the Platonic solids relate to the limit? How does it work? Does the visible universe push through the infinite like a ship through the ocean, and are the regular polyhedra the wake? Is the discrete manifold bashing against the continuous manifold like a subatomic particle in a cyclotron, and are the Platonic solids the little pieces spinning off in a bubble tank? Or is it like graphite dust on a kettle drum head, when sounding different notes causes the dust to dance in different standing-wave patterns? What is it? What's going on?

For about 10 years I watched the Platonic solids, hoping they would show me something about the structure of the universe. I put cubes inside dodecahedra, tetrahedra inside

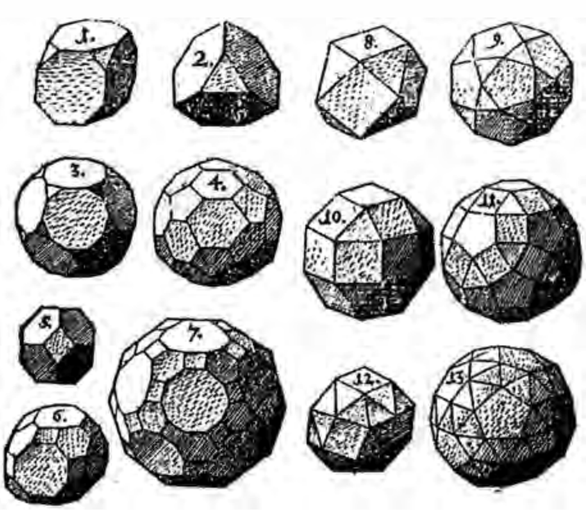


Figure 2
KEPLER'S ARCHIMEDEAN SOLIDS

Kepler did extensive studies of polyhedra, and made these drawings of the Archimedeans, which was part of his geometry tool chest.

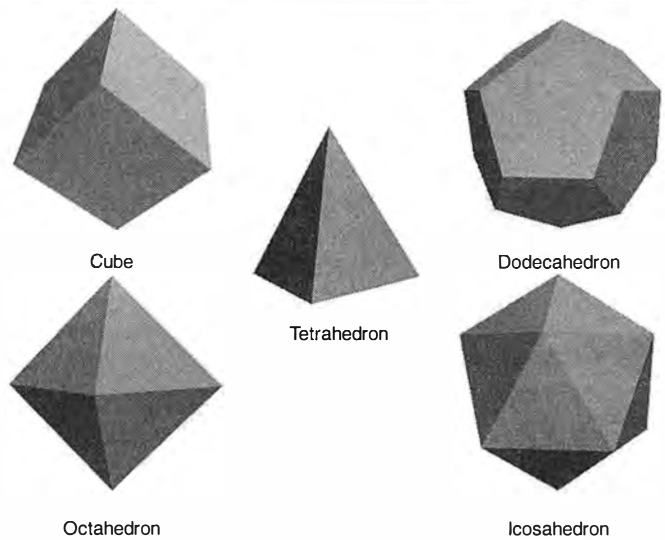


Figure 3
THE FIVE PLATONIC SOLIDS
The Platonic solids are convex polyhedra with regular, congruent faces whose edge-angles and vertices are equal. These are the only Platonic solid figures that can be constructed, hence the limit: cube, octahedron, tetrahedron, dodecahedron, and icosahedron. The cube and octahedron are dual-pairs, as are the dodecahedron and the icosahedron. Dual-pairs have switched numbers of faces and vertices, with identical numbers of edges.

cubes, octahedra inside tetrahedra; I paired duals, stellated those that would stellate, and sliced cubes and tetrahedra to see what their insides looked like. None of these "interrogation protocols" worked; they still wouldn't talk.

I knew about the Archimedean solids and didn't want to have anything to do with them. Compared to the nice, 5 Platonics, there were 13 Archimedeans, which is bad enough. Plus there was an infinite series of Archimedean prisms and *another* infinite series of Archimedean anti-prisms. And on top of that they all have duals, the Archimedeans, the prisms, and anti-prisms; and they are not duals of each other like the good old Platonic solids are, either. Each of the 13 Archimedean shapes has a unique dual that isn't an Archimedean solid, and all the prisms and anti-prisms have unique duals, too. Infinity times 4 plus 13 Archimedeans *twice* was too much. Archimedeans weren't for me. The 5 Platonics did their job; I could handle that just fine.

Spheres Were My Downfall

You can arrange each Platonic solid so that its vertices can touch the inside of one sphere. When you do that, it is said to be inscribed in the sphere. The center of each face of a Platonic can also touch another sphere. So can the center points of their edges. A different sphere can touch each location on each polyhedron. This comes from the regularity of the Platonic solids. Spheres are important because they represent least action in space. Just like a circle on a plane, spheres enclose the most area with the least surface.

Spheres represent the cause of the limit you run into when you try to make more than five Platonic solids. Just like the guy in Flatland,[2] who saw only a circle when a sphere popped into his world, the sphere is the highest level of least action we can apprehend with our senses alone. Perhaps the vertices of a Platonic solid don't define a sphere, but the sphere (or the nature of space that makes the sphere unique) is what limits the Platonics. That's more likely. Spheres are what the limit looks like to us if we're paying attention.

That's an important part of studying geometry. How does the infinite impact the universe we can see? Where does the complex domain intersect our domain? It's hard to see. The guy in Flatland looked at a circle and saw a line segment; never mind the sphere that created the circle that looked like a line to him. We aren't in much better shape than he was, when we are looking at spheres. Spheres, without the proper shading, just look like circles to us.

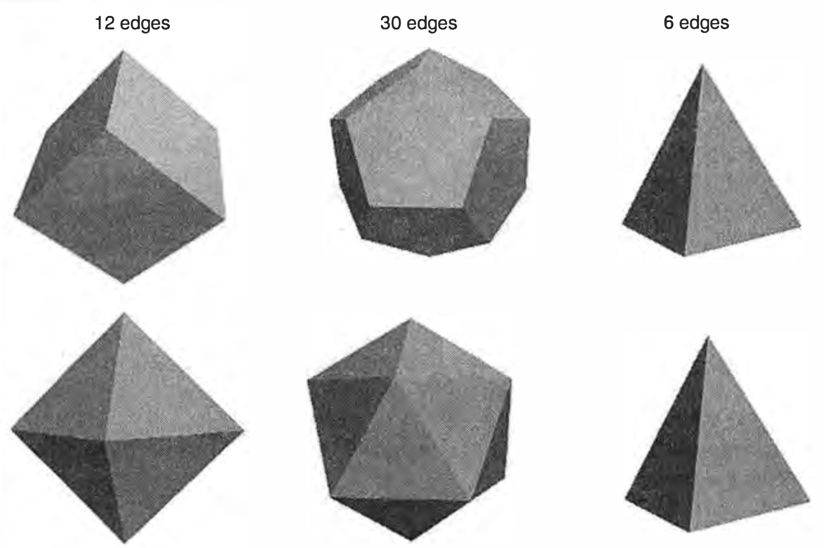


Figure 4
THE PLATONIC DUALS
Note that the tetrahedron is the dual of itself.

Spherical Geography

A straight line on a sphere is a great circle, like the equator of the Earth. Look at a globe; we are talking about geometry (Geo = Earth, metry = measure), right? Great circles are why Charles Lindbergh flew over Ireland to get to Paris. There are no parallel straight lines on a sphere. Any two great circles intersect each other, not once, but twice, at exactly opposite sides of the sphere.

You can do a neat trick with least action on a sphere. I saw this first in a videotape of a class given by Larry Hecht (editor-in-chief of *21st Century* magazine), and later Lyndon LaRouche featured the process in his paper "On the Subject of Metaphor" in *Fidelio* magazine.[3]

If you divide a great circle on a sphere with another great circle, they divide each other in half, as stated above. Picture the equator and what we laughingly call the Greenwich Meridian on Earth. Of course the two great circles don't have to be at right angles to each other; either of them can rotate around the points where they meet (in this case in the Gulf of Guinea, off Ghana, and in the Pacific Ocean, where the equator and International Date Line meet—Figure 8).

If you want to see how great circles divide each other in even divisions other than just in half, then the fun begins.

Take our original two great circles. Go to where they meet, off Africa, and move west on the equator until you hit the Galapagos Islands and stop. You are ready to create a third great circle. Turn right and go north. You zip over Guatemala, then over Minnesota, the North Pole, where you intersect the second great circle, Siberia, China, Indian Ocean, equator again, Antarctica, the South Pole is another intersection, South Pacific, and you are back where you started, having intersected the equator twice and the International Dateline/Greenwich great circle twice, too.

Now what do you have? The equator is divided into 4 equal parts by the other 2 great circles. So is the International Date Line great circle, and so is our new great circle. Three great circles dividing each other into 4 equal parts. The sphere of the Earth was just divided into 8 equilateral, right triangles by our 3 great circles (Figure 9). The great circles intersect at 6 locations. I wonder how many different ways you can divide great circles evenly with other great circles?

We got 4 even divisions with 3 great circles, how about 3 even divisions? Well, if you take the equator, or, I hope by now a 12-inch-diameter embroidery hoop, and divide it by other great circles into 3 parts, you don't get 3 parts. You get 6 parts, because pairs of great circles meet at opposite points of the sphere. There are no odd-numbered divisions of a great circle by other great circles. Let's see what these 4 great circles do. First, make sure the 3 great circles dividing your original one are also evenly divided into 6 segments by each other, and see what we have: All 4 circles are divided into 6 equal parts—spherical equilateral triangles alternating with spherical squares above and below the original circle, and triangles surrounding each pole.

Six squares and 8 triangles; does that sound familiar?

Anyway, we are about to hit a limit here, just to warn you. The only other way for great circles to evenly divide themselves on a sphere is with 6 of them dividing each other into 10 even segments. Try dividing one great circle into 5 equal parts—you can't do it; it will make 10 divisions, just like 3 forced 6. This is very hard to see if you haven't done it yourself—so, do it yourself. You can get a pair of 12-inch-diameter embroidery hoops for about a dollar. What you end up with is really pretty, too. It is a metaphor you can hold in your hand.

Twelve spherical pentagons and 20 spherical triangles. That sounds familiar too.

Three hoops, 4 hoops, and 6 hoops; and no other combination will evenly divide great circles—another limit, just like the

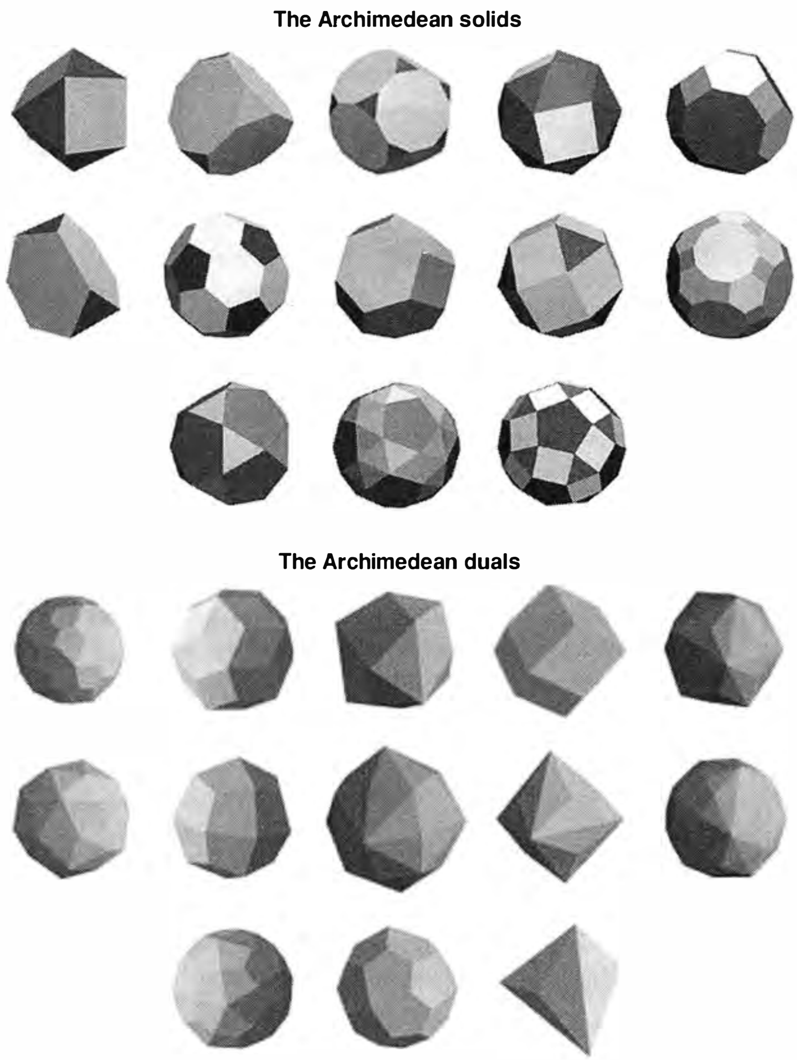


Figure 5
THE 13 ARCHIMEDEAN SOLIDS AND THEIR DUALS
All hell breaks loose. The polyhedral neighborhood becomes very crowded. There are 13 Archimedean polyhedra, and they all have duals.

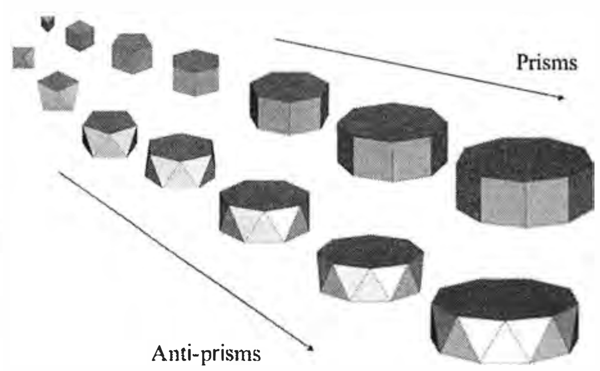


Figure 6
PRISMS AND ANTIPRISMS
There are infinite series of the prisms and the anti-prisms, and they all have duals too.

NASA

Figure 7
OUR EARTHLY SPHERE
The sphere represents least action in "three dimensions." A great circle is a straight line in spherical geometry.

Platonic solids are limited in number. (See Figure 9.)

But, this is the killer: Look at the 4-hoop construction. See the 12 places the hoops intersect each other? There are 6 around the middle, 3 on top and 3 on the bottom.

Well, if you stacked up identical marbles, you could put 6 marbles around one marble on a flat surface. Make sure that each of those 6 marbles have 6 around them, too. Keep doing this over and over, and cover your whole floor with a neatly arranged layer of marbles; then get ready for the second level. In the second layer, you could put 3 marbles around any one marble in the first layer, either in the 12, 4, and 8 o'clock positions, or alternately, in the 2, 6, and 10 o'clock positions. Choose one of the two arrangements and add enough marbles, and you will complete the second level, which will look just like the first level.

When it comes time to do the third level, you have a decision to make. You can put the third level in one of two orientations. You can put them directly over the marbles in the first level, or you can take the path less travelled: Put the marbles over the position you *didn't* select for the second level. If you do this, and keep the pattern up until you fill your room entirely with marbles, you will have two things, besides a heck-of-a-lot of marbles. One is a room filled with the *most* marbles that could possibly be put into the room, no matter what other method you used to stack them up: They are "close-packed." The other thing is this: Look at any marble. Where does it touch the other marbles? It touches 6 around the middle, 3 on top and 3 on the bottom—just like the intersections of the 4 hoops! The even divisions of 4 great circles generate the very same singularities where the hoops intersect, that close-packing of spheres does where the spheres touch. (See Figure 10.)

Remember that I didn't want to construct the Archimedean solids? Here's how it happened.

The spherical faces of the 4-hoop construction represent an Archimedean solid called the cuboctahedron: "Cube-octahedron" is 6 squares and 8 triangles. The dual of the cuboctahedron is called the rhombic dodecahedron. Dodecahedron means that it has 12 faces, like the regular Platonic dodecahedron; and rhombic means the faces are rhombic in shape, that is, diamond-shaped rather than the pentagonal shape you are used to. The rhombic dodecahedron is the shape of the honeycomb that Kepler discusses in the "The Six-Cornered Snowflake" paper.[4] Rhombic dodecahedra fill space. That means you can stack them up with no air between them. Because spheres close-pack in a way that generates the vertices of cuboctahedra, the dual of

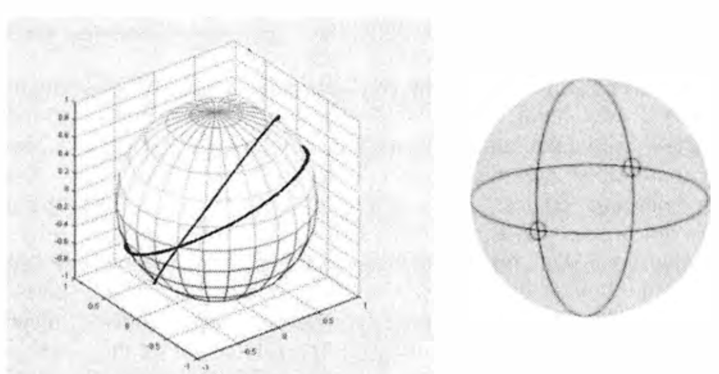


Figure 8
GREAT CIRCLES INTERSECTING
Great circles intersecting each other on a sphere always divide each other in half. That is about as "least action" as you can get.

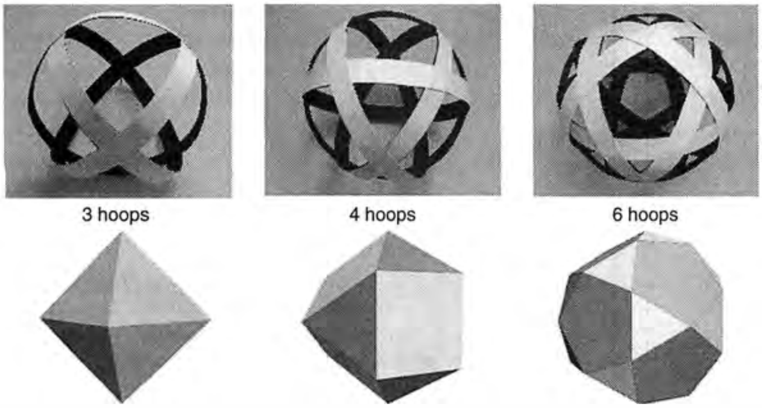


Figure 9
GREAT CIRCLES AND SPHERICAL POLYGONS
There are only three ways great circles can divide themselves into even sections which result in spherical polygons.The 3, 4, and 6 great-circle hoops represent the spheres which contain the great-circle figures below them: octahedron, cuboctahedron, and icosidodecahedron.

the cuboctahedron, by definition, can fill space. Now this wouldn't be so earthshaking, except for this fact: There is only one other polyhedron in the entire universe that has all of its faces identically shaped, and can fill space the way the rhombic dodecahedron does; that is the cube. Just those two with that limit—the cube and rhombic dodecahedron—and nothing else fills space.[5]

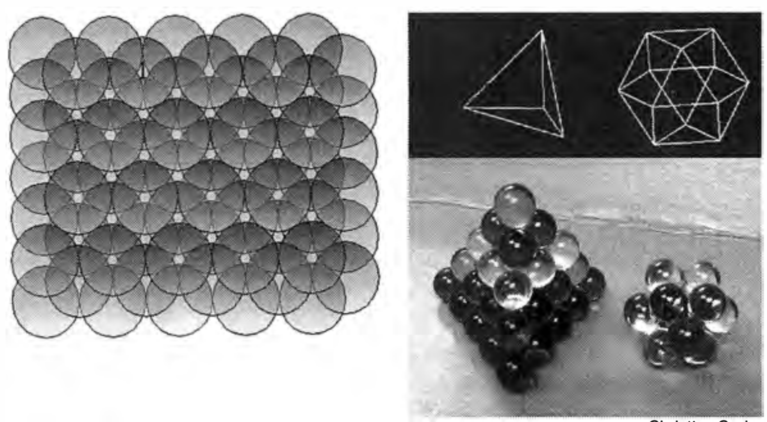
When Larry Hecht pointed this out on the videotape I saw, my heart sank. I knew that I was trapped; I had to construct the Archimedean solids,[6] because the dual of one of the Archimedeans had expressed a relationship to the same kind of limit that the Platonic solids express. This is the same limit that the great circles represent when evenly dividing themselves. It was all one package.

I was cornered. I felt like that old bastard Parmenides, who was trapped by the young Socrates into laboriously defending his life's work, rather than playing mind games with a group of bright young people. Socrates had accused Parmenides' henchman, Zeno, of lying to advance Parmenides' theories. Zeno and Parmenides responded not by losing their temper, but by trying to recruit Socrates to their way of thinking (the best defense is a good offense, even back then), but Socrates maneuvered *Zeno* into having Parmenides go through the *whole* thing. Parmenides didn't want to, and said, ". . . and so I seem to myself to fear, remembering how great a sea of words I must whirl about in."[7] Yes, I was caught.

What Most People Think Archimedean Solids Are

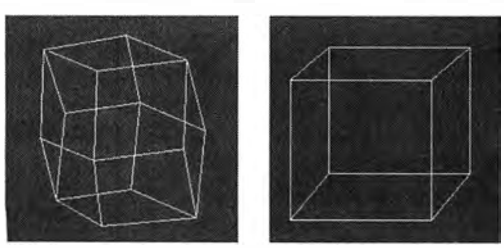
Here are the 13 different Archimedean shapes: Two of them, we are told, are more regular than the others, and are called "quasi-regular." You have already run into them; they are the cuboctahedron and the icosidodecahedron, which are defined by the 4- and 6-great-circle constructions. The cuboctahedron has the 6 square faces of the cube and the 8 triangular faces of the octahedron. The icosidodecahedron has 12 pentagonal faces like the dodecahedron and 20 triangular faces like the icosahedron. (See Figure 12.)

The next five of the Archimedeans are not a big problem to visualize either; I call them the truncated Platonic group (Figure 13). There is one of them for each Platonic solid, and they include the only polyhedron that people regularly kill and die for to this day, the truncated icosahedron, which is in the shape of a soccer ball.[8]



Christine Craig

Figure 10
CLOSEST PACKING IN SPHERES



Rhombic dodecahedron
Cube

Figure 11
SPACE-FILLING POLYHEDRA
A Platonic solid, the cube, and the dual of an Archimedean, the rhombic dodecahedron, are the only two space-filling polyhedra with identical faces. The rhombic dodecahedron is the shape of the honeycomb cells made by bees. Can you see the hexagons implicit in the figure?

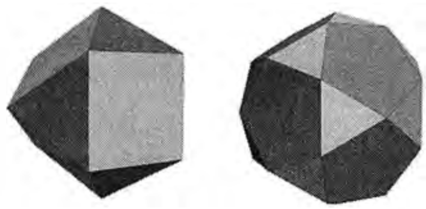


Figure 12
QUASI-REGULAR POLYHEDRA
The quasi-regular polyhedra are the great-circle figures containing the dual Platonic solids reflected in their names.

In each case you can imagine starting with a Platonic solid. For each Platonic face, however, there is a face with twice the number of sides. For example, the truncated cube has 6 octagonal faces instead of the 6 square faces of a cube. Where the Platonic solid had a vertex, there is now a face, which looks like the faces of the dual of the original Platonic solid. The truncated cube has 8 triangular faces, located where the cube's vertices were, situated in the same axis as the octahedron's faces. This works for the others, too. The truncated octahedron has 8 hexagonal faces and 6 square ones. The truncated tetrahedron has 4 hexagonal faces from the 4 triangles of the tetrahedron. The tetrahedron's dual is the same shape as itself, so you have 4 triangles in the truncated tetrahedron, too. The truncated dodecahedron has 12 ten-sided faces and 20 triangles, while the truncated icosahedron has 20 hexagons and 12 pentagons.

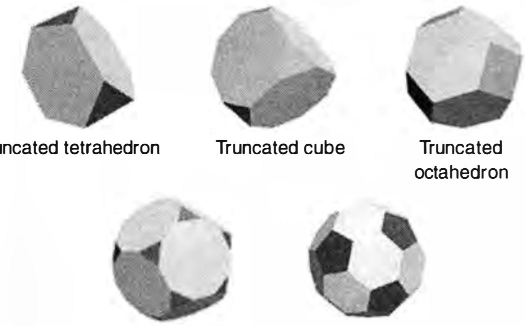


Figure 13
TRUNCATED PLATONIC GROUP
These are the Archimedean polyhedra which appear to result from truncation transformations on the corresponding Platonic solids.

That wasn't too bad. We are done with 7 out of 13 already.

It does get stranger from here on out, though. In ascending order of weirdness, you next have a pair of solids, which I call truncated quasi (quasi, for short) because they are truncated versions of the quasi-regular Archimedean solids. These are the truncated cuboctahedron and the truncated icosidodecahedron (Figure 14). Where the cuboctahedron has squares and triangles, the truncated cuboctahedron has octagons and hexagons. In addition, where the cuboctahedron has 12 vertices, the truncated cuboctahedron has 12 square faces. Where the icosidodecahedron has pentagons and triangles, the truncated icosidodecahedron has 10-sided faces and hexa-

Figure 14
TRUNCATED QUASI-REGULAR SOLIDS
The great-circle figures (the quasi-regular polyhedra) can also be truncated, giving the truncated cuboctahedron and the truncated icosidodecahedron.

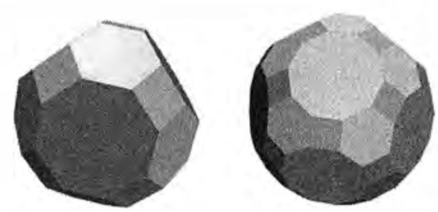
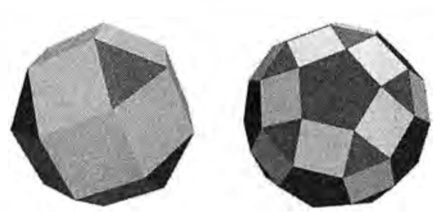


Figure 15
RHOMBIC QUASI-REGULAR SOLIDS
Transformations can also be made on the great-circle (quasi-regular) Archimedean polyhedra, leading to the rhombic great-circle figures, the rhombicuboctahedron and the rhombicosidodecahedron.



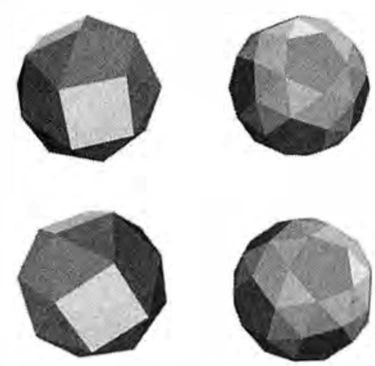


Figure 16
SNUB LEFT- AND RIGHT-HANDED
There are two sets of snub polyhedra in the standard Archimedean arrangement: the left- and right-handed snub cubes, and the left- and right-handed snub dodecahedra.

gons, with the addition of 30 square faces where the icosidodecahedron vertices were (Figure 14).

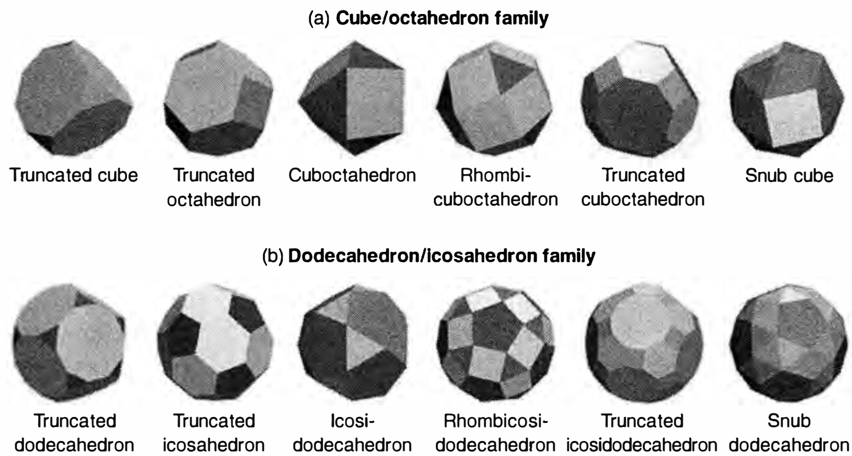
The next pair, the rhombicuboctahedron and the rhombicosidodecahedron, are simpler, but one of them is harder to see. These are called rhombi-quasi polyhedra, and they have the same square faces from the vertices of the quasi-regular solids as the previous pair does, but the other faces are the same shape as those of the quasi-regular solids, themselves, not double the number, like in the quasi, above. The rhombicosidodecahedron has 12 pentagons, 20 triangles and 30 squares for faces, and looks kind of obviously what it is, but the rhombicuboctahedron has 18 square and 8 triangular faces (Figure 15). This confused me when I first saw it, because the squares, even though they looked alike, actually came from two different processes (the square faces of the cube, and squares from the vertices of the cuboctahedron). This is the kind of ambiguity that can drive you nuts, until you realize that the whole point of what you are doing, in the geometry biz, is finding this kind of puzzle, and solving it.

Speaking of ambiguity that can drive you nuts, the last two Archimedeans are the snub cube and the snub dodecahedron. The snub cube, mercifully has 6 square faces. So far so good, but it also has 30 triangular faces. The snub dodecahedron has the expected 12 pentagonal faces, and *80* triangular faces. If you think that's bad, I'll tell you that there really are two different snub cubes and two different snub dodecahedra. They are made up of the same parts, but the way they are put together makes them look like they are twisted to either the left or the right (Figure 16).

That's it; those are the 13 Archimedean shapes.

The way these shapes are traditionally organized is apparent from their names. There are three sets arranged by dual-pair type: the tetrahedron, the cube/octahedron, and the dodecahedron/icosahedron. One set contains only the truncated tetrahedron. The next one contains the truncated cube and truncated octahedron, the cuboctahedron, the rhombicuboctahedron, the snub cube, and the truncated cuboctahedron. Finally, you have a set containing the truncated dodecahedron and truncated icosahedron, the icosidodecahedron, the rhombicosidodecahedron, the snub dodecahedron, and the truncated icosidodecahedron.

Now, I tried a more clever approach, asking why the tetrahedron group was such a little, nubby family, while the other Platonic solids have such nice big families?



(c) **Tetrahedron family**
Truncated tetrahedron, all by itself

Figure 17
STANDARD ARCHIMEDEAN ARRANGEMENT
Two families of polyhedra are related to the Platonics: the cube/octahedron family, and the dodecahedron/icosahedron family. The truncated tetrahedron sits alone in the tetrahedron family.

What Archimedean Polyhedra?

Act 1, scene 1 of *King Lear:*

REGAN: Sir, I am made
Of the self-same metal that my sister is,
And prize me at her worth. In my true heart
I find she names my very deed of love;
Only she comes too short. . . . [9]

Figure 18
THE TRUNCATED CUBE
The truncated cube has 6 octagonal faces where the cube had 6 square faces, and 8 triangular faces where the cube had 8 three-face vertices.

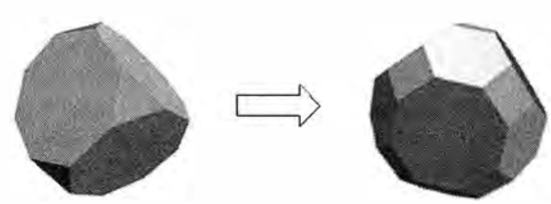


Figure 19
FROM THE TRUNCATED CUBE TO THE TRUNCATED CUBOCTAHEDRON
The truncated cuboctahedron retains the 6 octagonal faces from the truncated cube, but 8 hexagonal faces replace the 8 triangular faces. Additionally, 12 squares appear where the cube's edges were.

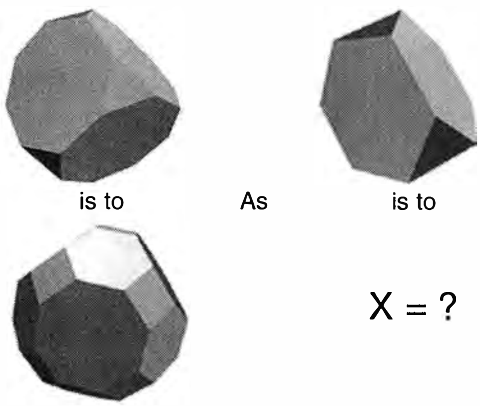


Figure 20
ANOTHER TRANSFORMATION TO PRODUCE ?
The truncated tetrahedron (upper right) has 4 hexagonal faces in place of the tetrahedron's triangular faces, and 4 triangular faces where the tetrahedron had vertices. Transform it analogously to the transformation of the truncated cube to the truncated cuboctahedron. Retain the 4 hexagonal faces from the truncated tetrahedron, and add 4 more hexagonal faces to replace its triangular faces, then add 6 squares, one for each tetrahedron's edge. What do you get?

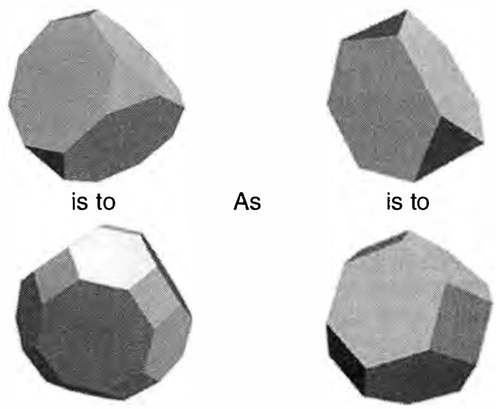


Figure 21
A TRUNCATED TETRITETRAHEDRON
Eight hexagons, 6 squares! A truncated tetritetrahedron—a new role for the Archimedean solid also known as the truncated octahedron.

After I saw Larry Hecht's class, I did make all the Archimedean solids. It took weeks, and I highly recommend that readers do the same. You can look at a still picture of them, or nowadays even download an interactive file from the internet, but it isn't the same as planning how many of each face you need, constructing the faces, and trying to fit them together so that it looks like it is supposed to. Anyway, in making the Archimedean solids, I became more and more upset at the injustice being meted out to our little friend, the tetrahedron. Not only did he have to pretend he had a dual by acting the part himself, but where the other Platonic dual-pairs have 6 or 7 Archimedean solids associated with them (if you count left- and right-handed snubs separately you get 7 each), the tetrahedron had only one Archimedean to play with.

I decided that this injustice would not stand. But what could I do about it? One thing I knew, I wasn't going to mess with the dodecahedron family—80 triangles in a snub dodecahedron? So, the cube family it is. The truncated cuboctahedron looked busy enough to get my teeth into, and the truncated cube looked to me like what was happening on it was clear enough, so that's where I started. I set up this puzzle: What would you get if you did to the truncated tetrahedron the same thing that was done to a truncated cube to get a truncated cuboctahedron? You know, *A* is to *B* as *C* is to *X*. What could be easier? (Figures18-21.)

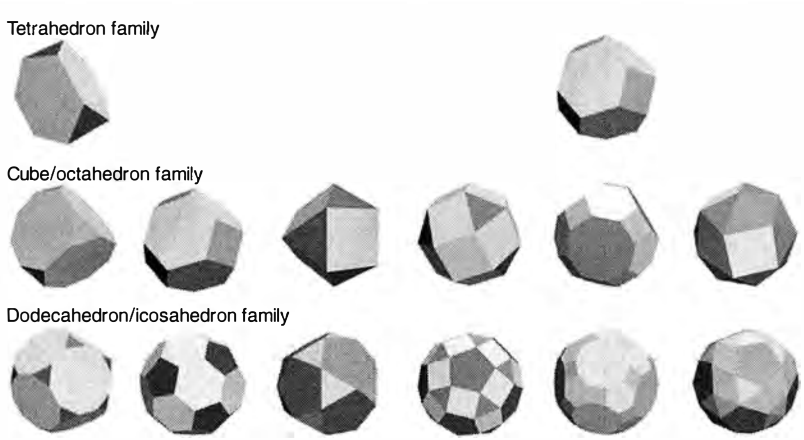


Figure 22
AN ADDITION TO THE ARCHIMEDEAN SOLIDS
The tetrahedron family now has two members.

The truncated cube has 6 octagonal faces, and so does the truncated cuboctahedron. The truncated cuboctahedron has 8 hexagonal faces where the truncated cube has 8 triangles. So far so good. And the truncated cuboctahedron has 12 square faces, where the cube has 12 edges. That is the *A* is to *B* part. Now for the "*C* is to *X*" part: The truncated tetrahedron has 4 hexagonal faces, so *X* has 4 hexagonal faces, too. Four triangular faces become 4 other hexagonal faces, and the 6 edges of a tetrahedron become 6 square faces in *X*. What is it? What do we have? Four plus 4 hexagonal faces are 8 hexagonal faces and 6 square faces. Eight hexagonal faces and 6 square faces; it has to work.

It does! Eureka! A new polyhedron lives! The tetrahedron has another family member. It's alive! I've invented a new Archimedean solid: 8 hexagons and 6 square faces, and it has all of its fingers and toes. It looks just like. . . .

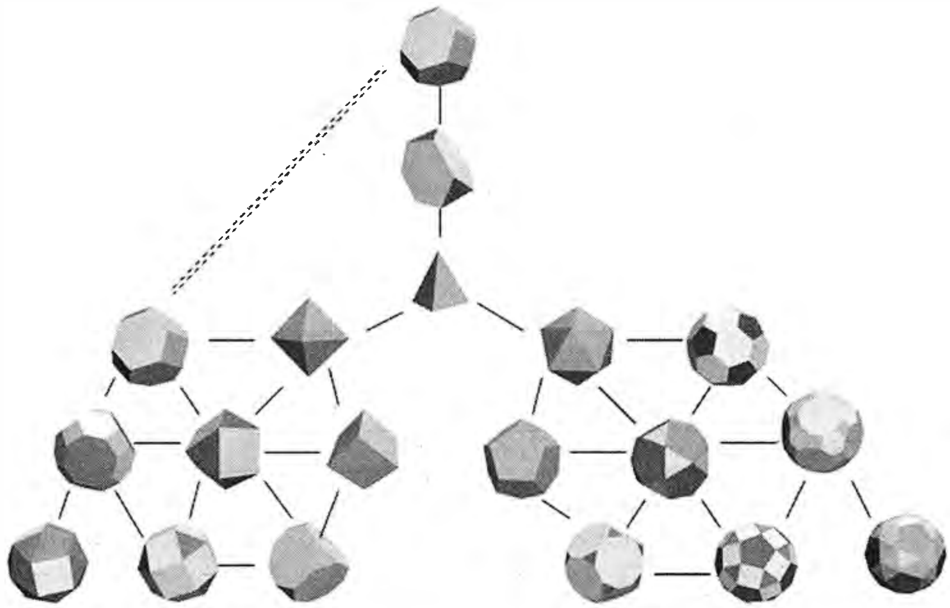


Figure 23
THE SHAPE OF SPACE
Another view of the Archimedean families.

Wait a Minute

What does it look just like? We have already *done* 8 hexagons and 6 squares, and if it is an Archimedean solid with regular faces, and all, then they both have to be the same shape: the truncated octahedron.

Yes, look at it, the truncated octahedron, 8 hexagons and 6 squares, is sitting in the tetrahedron family, acting like a truncated quasi, a *truncated tetritetrahedron.* The cube family is intersecting with the tetrahedron family. The shape of the truncated octahedron is acting like a truncated tetritetrahedron, just like F# on the piano is also G-flat. They are "enharmonic shapes."

When I first discovered this, I was so happy, I almost forgot entirely my mission of grilling the Platonic solids for their secrets. I made an attractive, nicely colored poster with the pretentious name, "The Shape of Space," which had the Platonic and Archimedean solids arranged in the symmetrical cube/octahedron and dodecahedron/icosahedron families, centered on the quasi-regular polyhedra; and the truncated tetritetrahedron was connected to the cube family with little dotted lines. It was pretty, and took some time to make, but completely ignored the fact that the tetrahedron still had a long way to go to achieve the equal rights it deserves as a fully vested Archimedean solid and head of a family.

At that point, LaRouche put out his "Metaphor" paper, in which he hit the great-circle question really hard. The "Metaphor" paper set me to thinking again. I had supposed that the sphere had to be a major waypoint on the route to the creation of the Platonic solids; and the quasi-regular solids (the cuboctahedron and icosidodecahedron) were clearly generated by even divisions of great circles on a sphere; and LaRouche made no bones about the fact that *the* way to construct the Platonic solids was with great circles on spheres. But why, then, was the epitome of clean, least action resulting in an oddball hodgepodge of two Archimedean solids and one Platonic solid? (Figures 24.)

This was really messy. When I first wrote about this 10 years ago I said, "How anomalous." What I meant to say was, "Is the Composer of the universe a spaz?" Who would design something that odd?

What bothered me was the apparent unevenness of the pattern in my shape-of-space chart. It was that tetrahedron family that was out of place. I *finally* decided to look in that direction.

I knew that while the cube was the dual of the octahedron, and the dodecahedron was the dual of the icosahedron, the tetrahedron was the dual of itself. Well, in order to examine

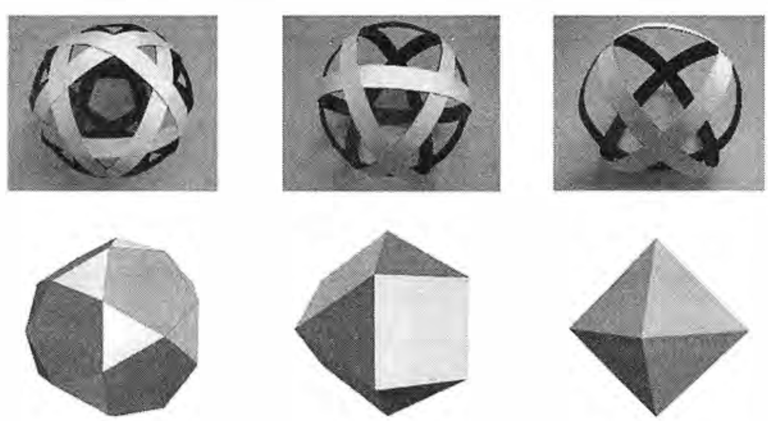


Figure 24
EVEN DIVISIONS OF GREAT CIRCLES
The epitome of clean, least action—even divisions of great circles on a sphere—results in the oddball hodgepodge of two Archimedean solids and one Platonic solid.

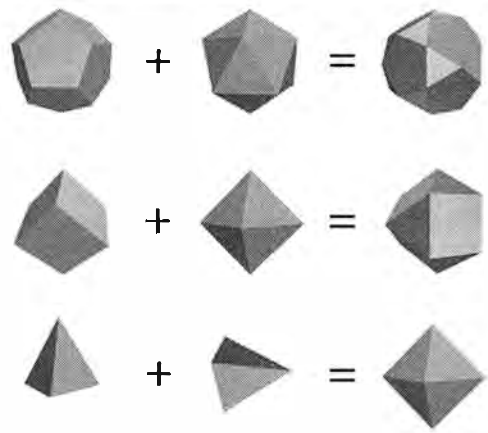


Figure 25
THE GREAT CIRCLE FIGURES AND DUAL PAIRS
Two of the polyhedra formed by great circles have faces reflecting the dual pairs of Platonic solids. The 12 pentagonal faces of the dodecahedron and the 20 triangular faces of the icosahedron create an icosidodecahedron. The 6 square faces of the cube and the 8 triangular faces of the octahedron create a cuboctahedron.

Because the tetrahedron is the dual of itself, you could say that there is a dual-pair of tetrahedra, too. You take the four triangular faces of one tetrahedron and the four triangular faces of the other tetrahedron and create—a tetritetrahedron, also known as an octahedron.

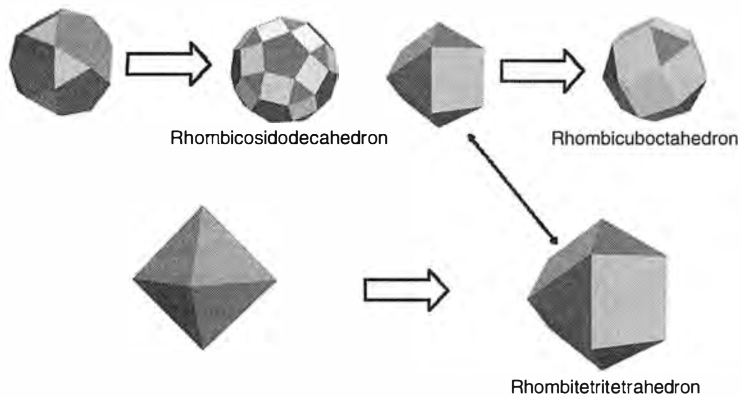


Figure 26
THE RHOMBIC TRANSFORMATIONS
In the cube/octahedron family, the rhombicuboctahedron can be formed by taking the cuboctahedron and adding squares in place of all its vertices. In the dodecahedron/icosahedron family, the rhombicosidodecahedron can be formed by taking the icosidodecahedron and adding squares in place of all its vertices.

Similarly, in the tetrahedron family you would start with the octahedron (or as we would call it in this family, the tetritetrahedron) and add squares to the vertices. That gives us the cuboctahedron again, known as the rhombitetritetrahedron in this enharmonic incarnation.

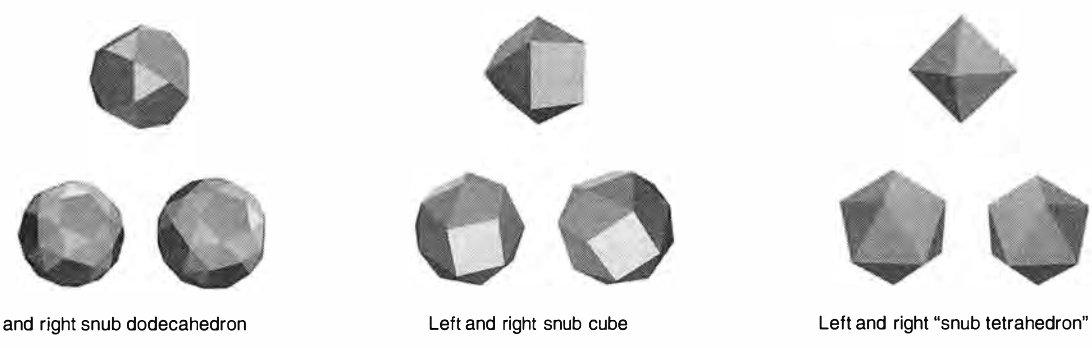
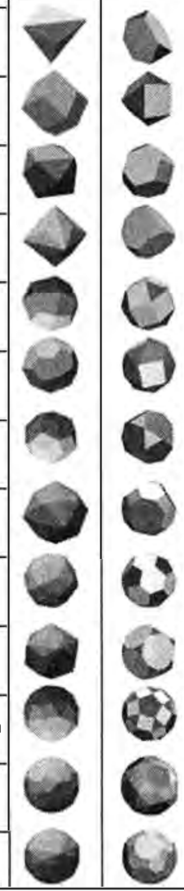


Figure 27
THE SNUB TRANSFORMATIONS
To make a snub cube, surround the square faces of a cube with an alternating lattice of triangles, with one triangle for each edge of each of the cube's faces, and one triangle for each face of the cube's dual, the octahedron.

To transform the great-circle (quasi-regular) icosidodecahedron to its snub, add 60 more triangles to the 12 pentagonal faces of the dodecahedron and the 20 triangles of the icosahedron—2 triangles for each of the icosidodecahedron edges.

And for a snub tetrahedron, take 4 triangles for the tetrahedron's faces, 4 triangles for the other tetrahedron's faces, and 12 triangles. That's 20 triangles, 2 for each of the tetritetrahedron (octahedron) edges—left- and right-handed, of course. Yet another enharmonic solid is revealed—the icosahedron—known in this relationship as the snub tetrahedron.



Vertices	Faces	Edges	Polyhedron	Polyhedron	Vertices	Faces	Edges
8	12	18	Triakis tetrahedron	Truncated tetrahedron (both of them)	12	8	18
14	12	24	Rhombic dodecahedron	Cuboctahedron	12	14	24
14	24	36	Tetrakis hexahedron	Truncated octahedron	24	14	36
14	24	36	Triakis octahedron	Truncated cube	24	14	36
26	24	48	Deltoidal icositetrahedron	Rhombicuboctahedron	24	26	48
38	24	60	Pentagonal icositetrahedron	Snub cube	24	38	60
32	30	60	Rhombic triacontahedron	Icosidodecahedron	30	32	60
26	48	72	Disdyakis dodecahedron	Truncated cuboctahedron	48	26	72
32	60	90	Pentakis dodecahedron	Truncated icosahedron	60	32	90
32	60	90	Triakis icosahedron	Truncated dodecahedron	60	32	90
62	60	120	Deltoidal hexecontahedron	Rhombicosidodecahedron	60	62	120
92	60	150	Pentagonal hexecontahedron	Snub dodecahedron	60	92	150
62	120	180	Disdyakis triacontahedron	Truncated icosidodecahedron	120	62	180

Figure 28
THE ARCHIMEDEANS AND THEIR DUALS

these anomalies, I decided to see if the tetrahedron family could be made to conform to the pattern created by the other two families.

If the tetrahedron is the dual of itself, then the truncated tetrahedron should show up in the pattern twice also. That makes sense.

The quasi-regular slot in the other families could be thought of as forming thus: Take the 6 square faces of the cube and the 8 triangular faces of the octahedron, and create a cuboctahedron. Take the 12 pentagonal faces of the dodecahedron and the 20 triangular faces of the icosahedron and create an icosidodecahedron (Figure 25). So, in the tetrahedron family you take the 4 triangular faces of the tetrahedron and the 4 triangular faces of the *other* tetrahedron and create. . . . The quasi-regular polyhedron in my hypothesized tetrahedron family was the octahedron, the very same figure that I had constructed for that slot using LaRouche's great-circle method earlier. That was amazing, even electrifying.

In an instant I went from a perception of a cluttered universe and a nice tidy theory, to a more orderly universe and a pet theory blown to smithereens.

Now I was sure I could fill up the empty spaces in the tetrahedron family. I only had two left to do. The rhombicuboctahedron looks like it is formed by taking the cuboctahedron and adding squares where the edges were (Figure 26). The rhombicosidodecahedron looks like you take the icosidodecahedron and add squares where its edges were. In the tetrahedron family you would start with the octahedron (or as we would now call it in this family, the tetritetrahedron) and add squares to the edges. What do you get? The result was a figure with 8 triangles and 6 squares—a cuboctahedron—a polyhedron already created, which we could now call the rhombitetritetrahedron, in this new, enharmonic incarnation.

This was getting interesting. I now had three polyhedra from the cuboctahedron family serving double-duty in the tetritetrahedron family, and there was one figure left: the "snub tetrahedron," if there were such a thing. Snubs (the snub cube and the snub dodecahedron) weren't on my "favorites" list. They were messy; they didn't have the same number of faces that the rest of their families did. The snub cube had 6 squares, all right, but had 32 triangles! The snub dodecahedron had the expected 12 pentagons, but 80 triangles, as already mentioned, and it wasn't clear what they all

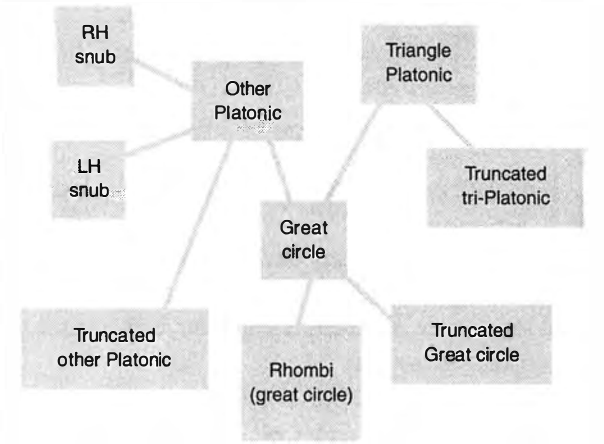


Figure 29
THE POLYHEDRAL FAMILY RELATIONSHIPS
This is one way of looking at the family relationships within any of the polyhedral families.

were doing or why. This was about the last time that an anomaly like that irritated me. I started to look forward to them after I did the work represented by the next paragraphs.

To make a snub cube, you surround the square faces of a cube with an alternating lattice of triangles. You have one triangle for each edge of *each* of the cube's faces, and one triangle for each face of the cube's dual, the octahedron. Six square faces and 6 times 4 sides is 24 triangles, plus 8 octahedral triangles makes the 32 triangles (Figure 27).

Likewise, in the snub dodecahedron you surround the pentagons in the same manner. Now, to create the supposed snub tetrahedron you would surround 4 triangles with the same pattern of alternating triangles. That is, 4 faces with 3 edges each, which would give you 12 triangles; add 4 triangles from the tetrahedron and 4 triangles from its dual. That would give you a figure made up of 12 plus 4 plus 4: 20 triangles. Do we have something like that already? Yes, of course we have 20; it's called the icosahedron! The icosahedron is also a snub tetrahedron, and the icosahedron is from the dodecahedron family, too, not the cube family. The dodecahedron family is enharmonically participating in the tetrahedron family, as well! All of a sudden, the snubs didn't seem so bad after all. They had filled up the tetrahedron family. The pattern was complete.

We now have three totally symmetrical families of polyhedra. Each family has the same number of members as the other two families, performing the same function in each family. Starting with even divisions of great circles on a sphere, with the 3, 4, and 6 hoops; each family has a polyhedron directly mapped from the vertices of the hoops. Every family also has two Platonics, duals of each other, whose faces are contained in the previous figure. They have a truncated version of each Platonic, a rhombic version of the great-circle figure, a truncated version of the great-circle figure, and a snub figure, left- and right-handed. The families are connected by three polyhedra in the cuboctahedron family and one member of the icosidodecahedron family, appearing in the tetrahedron family as "enharmonic" solids.

This was a milestone, but I wasn't done. One huge batch of work I foresaw was, how do you arrange the families so that both their symmetry *and* their interconnections are clear? That would be an updated and more accurate version of my old "Shape of Space" poster.

The other issue that came up some time later, as a surprise, was that each of the Archimedeans has a dual. How do they fit into the pattern?

Another big issue was this: Clearly, the Composer of the universe didn't hack off the vertices of a cube with a knife to make a truncated cube. How directly do great circles participate in the construction of the Archimedeans, or Platonics for that matter?

Where Archimedean Polyhedra Meet

We began with the assumption that space wasn't just an endless checkerboard. In investigating the limits of visible space, starting with the Platonic solids as symbolic of shapes that were formed by the confines built into the nature of creation, we fashioned a set of three, symmetrically ordered families of polyhedra, each containing Platonic and Archimedean solids.

The families are connected by three polyhedra shared by both the cuboctahedron and the tetrahedron families as enharmonic shapes. These are polyhedra that look alike, but whose genesis and usage in this scheme, make them different. There is also one member of the icosidodecahedron family that is enharmonically shared with the tetrahedron family as well. No member of the cube or dodecahedron family touches each other, but both of those families touch the tetrahedron family.

The significance of this arrangement goes back to the age-old appreciation of the uniqueness of the Platonic solids. The limit built into the universe is manifested in the fact that you can construct only five shapes that conform to the restrictions that define the Platonic solids. That same limit restricts the number of ways that the great circles divide each other evenly. There are only three ways to do it. Once you recognize the way the families intersect, you realize that you are looking at three symmetrical families, which contain three pairs of Platonic solids, generated by three sets of great-circle figures.[10]

After I remanufactured all the Platonic and Archimedean solids with the faces of each solid instructively colored, I wanted to develop a pedagogy that would enable people to see both the symmetry of the families and how they intersected. My set of all these polyhedra had the cube, and all faces of other polyhedra that shared the cube's orientation and function, colored green. The octahedra and its kin were yellow. One tetrahedron was red, with its dual orange. The dodecahedron and its co-functionaries were dark blue, and the icosahedron was light blue. The faces which represented variations on the vertices of the great-circle polyhedra, were colored white, black, or gray, depending on how many sides the faces of their Archimedean duals have. This arrangement showed the symmetry of the families brilliantly, but left the intersections of the families up to the imagination.

My first attempt to rectify this shortcoming looked like a model of a molecule—a rather alarming molecule, at that (Figure 29). A ring of 6 spheres represented the members of each family. These spheres represented the Platonic solids,

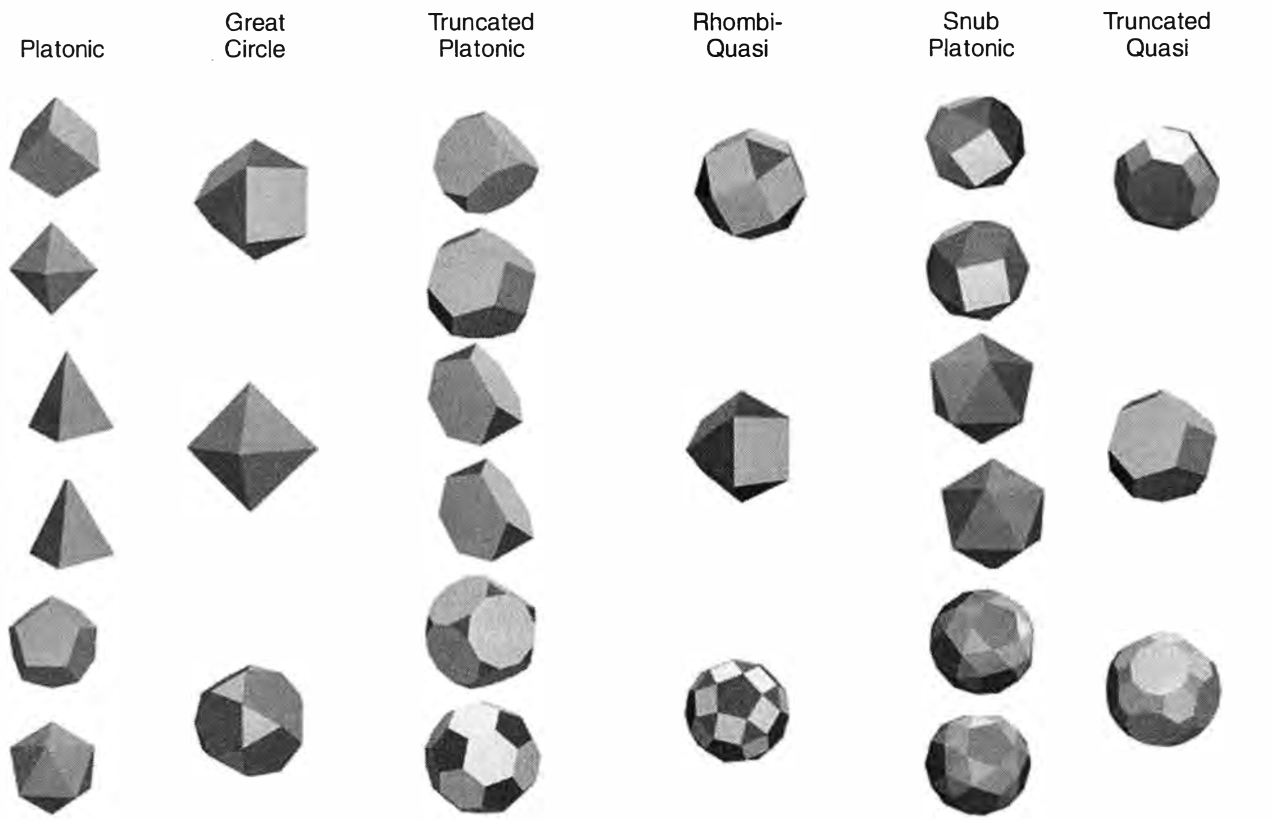


Figure 30
THE COMPLETE SET OF ARCHIMEDEANS
This is one way of looking at the family relationships within any of the polyhedral families.

both truncated Platonic solids, the truncated great circle, and the rhombic great-circle figure, all arranged around the great-circle figure itself. There was a tail attached at one Platonic, representing the snub figures. I later refined this arrangement to one that looked like one of a set of jacks: 6 balls, one above, one below, up, down, left, and right of the central ball, with one hanging off to the side.

I actually made three of these sets out of Styrofoam balls and toothpicks, and attached them to each other in the appropriate manner. If you did it just right, you could join the three families where they intersect, indicating the connections made by the enharmonic solids, with the octahedron touching the tetritetrahedron, the cuboctahedron touching the rhombitetritetrahedron, the truncated octahedron touching the truncated tetritetrahedron, and finally the icosahedron touching the snub tetrahedron.

I did it, but it was a mess. It was very hard to keep the construction from falling apart. And even when it held together (though it accurately represented what I wanted to show), you couldn't really see it. It had a decided Rube Goldberg quality.

This wasn't what I wanted at all. You had the sheer beauty of great circles on a sphere: Least-action pathways on a least-action surface, dividing themselves evenly and creating symmetrical families of polyhedra, which intersected in an ironical way, typifying the kind of certainty you can only find embedded in a metaphor, which, of course, is the only way to speak the truth. This truth represented a visible image of the unseen limits placed on physical space by the creating force of the universe. I didn't think a pile of crumbling Styrofoam was the right way to show this. I was stuck at this point for some days. Then I had an idea; I decided to display this irony ironically.

The irony was this: The unseen, uncreated domain, which bounds and is creating our universe, has limited our ability to create regular polyhedra and, as stated, proved that the universe is not shaped like an endless checkerboard. How to show this? Put it on a checkerboard.

Do What?

This really cheered me up. In discussing these polyhedra you have three attributes to contemplate, their faces, the edges where two faces meet, and the vertices where the edges and faces meet. For example, the tetrahedron has 4 faces, 4 vertices and 6 edges; the cube, 6 faces, 8 vertices and 12 edges. The reason the octahedron is the dual of the cube is that the octahedron has 8 faces where the cube has 8 vertices, 6 vertices where the cube has 6 faces, and 12 edges, which cross the cube's 12 edges at right angles. You get the idea. To map the polyhedral families,

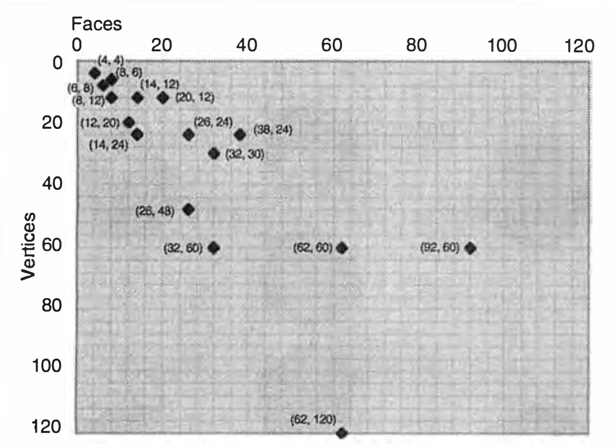


Figure 31
TWO-DIMENSIONAL GRID OF PLATONIC AND ARCHIMEDEAN POLYHEDRA

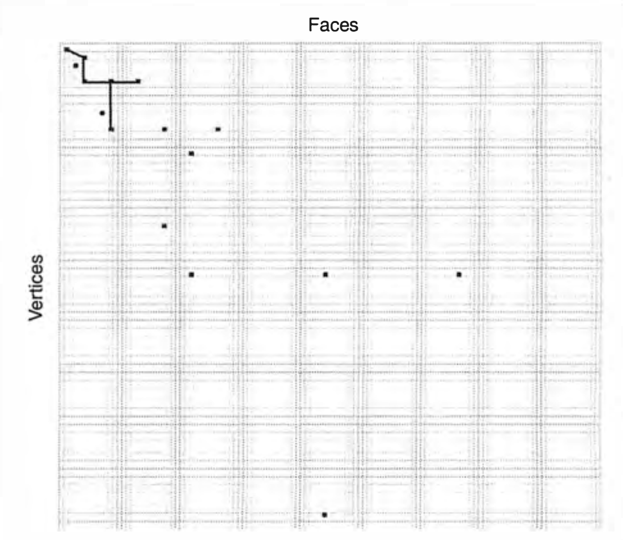


Figure 32
THE TETRAHEDRON FAMILY HIGHLIGHTED
The connected dots represent the tetrahedron family, masquerading as a small Deinonychus *dinosaur.*

find the location for each member on a three-dimensional grid, where each axis of the grid represents one of the attributes of the polyhedron: faces, edges, and vertices.

Since I was working with graph paper on a clipboard, I started by using only two axes at a time. I found it most effective to examine the faces and vertices on the two-dimensional graph paper and just ignore the edge-axis. (There is another irony here that took me years to understand, but no shortcuts). What I found at the time was really something. (See Figure 31)

I put the dots on the graph paper. It looked like a confusing mess, but when I connected each family's dots with colored ink, its clarity almost jumped off the paper. It looked like a star chart with constellations drawn on it. The constellation of each family looked like a primitive cave painting of a bird—a crane or pelican—or better yet, a theropod[11] dinosaur, one that looks like the *Tyrannosaurus rex*. The Platonics were located at the tip of each dinosaur's mouth; the great-circle figures were the heads and the truncated Platonics were the little front claws. The rhombic great circles were the bodies, the snubs the tips of the tails, and the truncated great-circle figures were the feet.

I had a "little" 8-foot-long, red *Deinonychus* dinosaur,[12] with its mouth closed representing the tetrahedron family; a medium-sized 16-foot, green *Ceratosaurus*[13] with its mouth open a little as the cube family, and a huge blue 40-foot-long *T-Rex*[14] with its mouth open wide, as the representative of the dodecahedron family. This was a lot of fun.

One thing that seemed funny to me was that the "truncated Platonic" pairs—the truncated cube and truncated octahedron, for example—both mapped to the same place, even though they had very different appearances. The same thing happened with the truncated dodecahedron and truncated icosahedron. Look at the truncated cube and truncated octahedron, or even more striking, the truncated dodecahedron and truncated icosahedron. They don't look at all alike, but each pair happens to have the same number of faces, vertices, and edges. Well, one polyhedron for each dinosaur claw.

You could see each family clearly on the chart, and the intersections, too: the tip of the mouth of the green *Ceratosaurus* touched the head of the red *Deinonychus*; the head of the *Ceratosaurus* touched the body of the *Deinonychus*; and the neck of the *Ceratosaurus* touched the foot of the *Deinonychus*. At the same time, the mouth of the blue *T-Rex* touched the tip of the tail of the poor little *Deinonychus*. This really worked nicely, and it gave you the impression that you weren't looking at a static thing. Those dinosaurs were going to start chewing any minute. You could also see how the enharmonic polyhedra were, in fact, in both families, filling different roles.

The dinosaur mouths were open different amounts. That made me stop and look. It seemed to mess up the symmetry of the families. I knew something was funny with the way I was thinking about this, and I had a glimmer of anticipation, like the change in the way the air feels before a thunderstorm. *Why weren't my supposedly symmetrical families absolutely identical on the chart?*

I had an idea—superimpose the families to see if they really were the same shape. They looked the same, but, you never know. Here's how it works: The vertices of the dodecahedral Archimedeans were at 30, 60, and 120; the cubic Archimedean vertices were at 12, 24, and 48; and the tetrahedral vertices were at 6, 12, and 24. All I had to do was put the dots on one grid that had three different scales. If the families were symmetrical, then the dots would be in the same place. The differences in dodecahedral Archimedean vertices were 30 and 60; the differences for the cubes were 12 and 24, with the tetrahedrons at 6 and 12. That should work.

The scale for the faces of the Archimedean polyhedra was the same idea. The dodecahedral Archimedean faces fell at 32, 62, and 92. The cubes were 14, 26, and 38; with the tetrahedrons at 8, 14, and 20. This worked too, with differences of

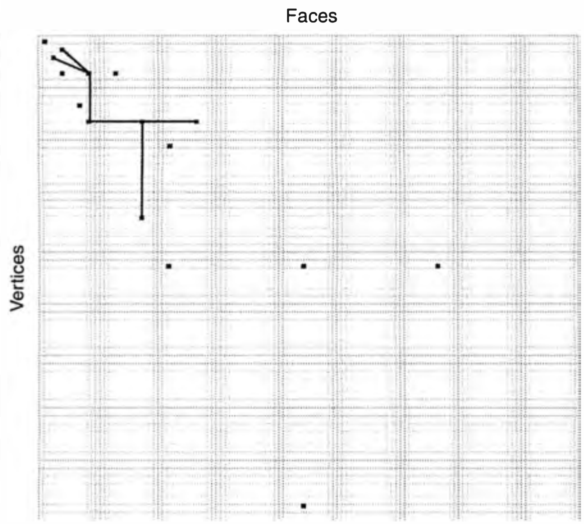


Figure 33
THE CUBE FAMILY HIGHLIGHTED
The cube family resembles a mid-sized Ceratosaurus *when its dots are connected.*

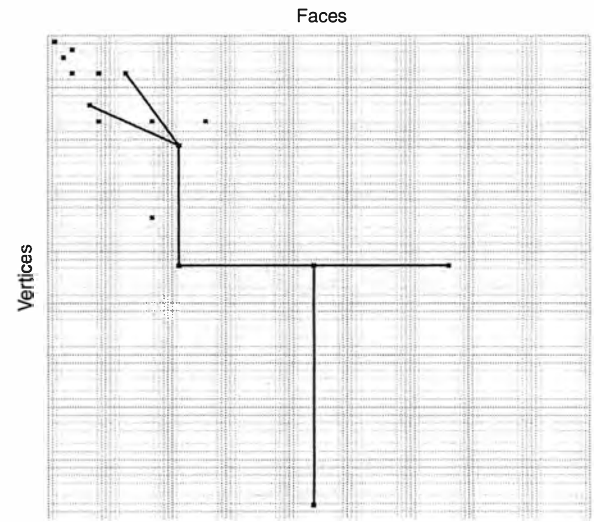


Figure 34
THE DODECAHEDRON FAMILY HIGHLIGHTED
The dodecahedron family as a huge T-rex.

30 for the dodecahedral, 12 for the cubic, and 6 for the tetrahedral. The Archimedean solids of each family exactly mapped onto the Archimedeans of all the other families. The Archimedean polyhedra families really were symmetrical.

One little nagging, hint of a question. The cube, dodecahedron, and tetrahedron all mapped onto each other too, when I overlaid the families of Archimedeans, but the octahedron and icosahedron each fell in a different place. That was why the mouths of the dinosaurs seemed to be open wider, as they got bigger. The bigger the dinosaur, the wider the mouth—maybe that had something to do with Darwin, but I doubted it. It was a puzzle, but by this time I was working so fast that I didn't stop.

This was a situation I was used to by now, in the geometry work. I had a nice theory, a beautiful picture to show, and one fly in the ointment. I found that you don't have to ruthlessly hunt down the anomalies and destroy them. Believe me, if you do the work, they'll find you. (What you do have to do is enjoy being caught by the anomalies, unlike the "Bread Scholars" that Schiller denounces, who try to cover up anomalies.[15])

Why weren't my symmetrical families symmetrical? Those damn dinosaurs had their mouths open different widths. I will tell you why, but we are going to have to go around the long way to get there.

Three Dimensions, If You Got 'Em

I did feel a little bad to be working with only two dimensions of my three-dimensional grid at one time. So, I got a slab of Styrofoam and some small wooden dowel-rods. I made a face- and vertex-grid on a piece of paper, cut the dowels to the length of the edge-axis on the same scale plus an inch, put the paper on the Styrofoam, and poked the dowels through the paper at the proper place an inch into the Styrofoam. The upper ends of the dowels represented the location in 3-D where the polyhedra should be located. I was so happy with this that I made a piece of cardboard which had pictures of each Platonic and Archimedean polyhedron on it. The cardboard would sit on the Styrofoam, next to where the dowels were, so you could see what each dowel represented.

I had hoped that looking at the pattern in three dimensions would directly portray some neat secret about the unseen force that shapes the Platonic and Archimedean solids. Maybe it would be a 3-D spiral, or waveform, or some exotic shape like a pseudosphere.

It didn't.

It looked to me like all the polyhedra fell in one plane, a plane tilted with respect to the other axes, but just a plane! Upon reflection, this shouldn't have been a surprise, if I had had more mathematical training. The phenomenon was an artifact of what has been sadly named Euler's formula. Each of the polyhedra is subject to this curious fact: The number of faces, plus the number of vertices, minus the number of edges is always 2.

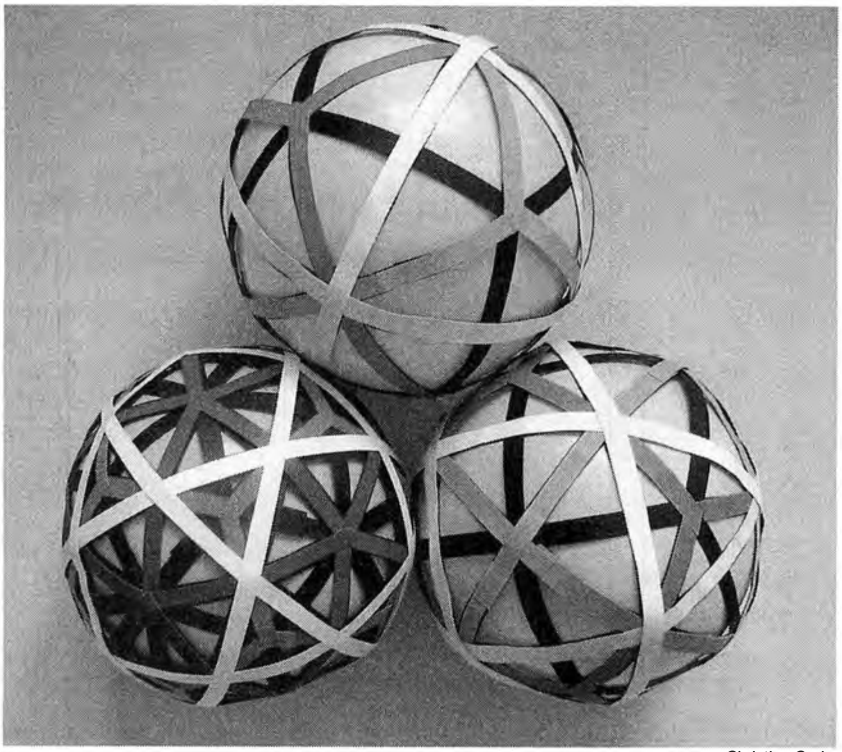
Tetrahedron: $4 + 4 - 6 = 2$.

Snub dodecahedron: $92 + 60 - 150 = 2$, and so on.

This would explain why all the solids, mapped the way I was doing it, ended up in a plane. It did make it easier to show. I could still accurately display the real three-dimensional graph on a two-dimensional piece of paper after all, but it lacked the pizzazz of having the more trendy hyperbolic waveforms in my graph.

'The Universe, and All That Surrounds It'[16]

In LaRouche's "Metaphor" paper, which was published when he was in prison, at the height of my activity in these matters, he made it quite clear that great circles on a sphere were the way to create the Platonic solids. My one overriding thought while working on this project was, "Spheres are primary; how does



Christine Craig

Figure 35
GOD'S GRAPH PAPER
The 6-, 9-, and 15-great-circle spheres, with the fundamental 3-, 4-, and 6-great circles of the Platonics superimposed on them. These are made of half-inch strips of colored poster board glued into the great circles. White balloons were inflated within to enhance visibility.

this come from a sphere?" A sphere is the highest level of least action we can apprehend with our senses alone.

The regular 6-hoop sphere, the icosidodecasphere, has 12 pentagonal and 20 triangular areas that the great circles sweep out. To locate a dodecahedron in this arrangement, you put each of its 20 vertices in the center of one of the 20 spherical triangles of the icosidodecasphere. Likewise, the icosahedron's 12 vertices would go into the 12 spherical pentagons of the icosidodecasphere. If you look at a dodecahedron alone, you see that it is like every other polyhedron we are dealing with, except the tetrahedron, in this way: It is made up of features that reappear on opposite sides of the figure. Each face has a parallel face that is on the other side of *the* dodecahedron, so a dodecahedron is really made up of 6 pairs of parallel faces. Likewise, the vertices all have another vertex exactly opposite to it on the other side of the dodecahedron. The edges do too. Look at the 30 edges of the dodecahedron. If we imagine the dodecahedron inside a sphere that touches each of its vertices and imagine a segment of a great circle connecting each vertex to form a dodecasphere, then we are ready for action.

Take any edge on the spherical dodecahedron, the dodecasphere. This is a segment of a great circle. Extend the segment in a straight line on the sphere. The line (great-circle path) cuts through the center of the next face, cuts a different edge in half at right angles, cuts through the center of another face and then joins up with the edge on the opposite side of the dodecahedron. It continues on until it returns to the original great-circle segment. If you can see it (it is really hard), you will find that it takes 1 great circle to cover 2 edges of the dodecahedron. Since there are 30 edges on a dodecahedron, it takes 15 great circles to define a dodecahedron.

Fifteen great circles! I can barely see 4 great circles when I'm looking right at them. How can I visualize 15?

Remember the Bread Scholars? You have to do it. For safety's sake, don't use 15 embroidery hoops for this. Use a half-inch strip cut the long way from a piece of poster board. Mark the strips where they will intersect before you cut them out. There is a lot of technique involved in getting them to work, but that's part of the fun, too.

Remember the dodecahedron inside the icosidodecasphere? The center of each of the dodecahedral edges touches a vertex of the icosidodecasphere. There are 30 edges to a dodecahedron, and 30 vertices in an icosidodecahedron, and they are, indeed, in the same orientation. Because that's true, look at where the 15 great circles go. They all bisect the vertices of the icosidodecasphere, clean as a whistle.

Look at an icosahedron inside an icosidodecasphere. Remember that? 12 vertices are inside 12 spherical pentagons. The center point of the each of the icosahedral edges touches each icosidodecasphere at the vertex—30 and 30, its just like the dodecahedron. The 30 edges of an icosidodecahedron would make 15 great circles, just like the dodecahedron did. In fact they are the very same 15 great circles.

Now look at this process backwards. You start with a sphere—least action in the visible domain. Straight lines on the sphere, great circles, intersect each other to give you even divisions. This can be done in only three ways, with 3, 4, and 6 great circles. Take the 6-great-circle sphere, the icosidodecasphere, and bisect each angle where the 6 great circles meet at each vertex with another great circle. These 15 great circles have created the vertices of both the dodecahedron and the icosahedron. You have done it: least action, to spheres, to Platonic solids.

Now, you could slice up a dodecahedron to make the other Platonic solids without using the other regular great-circle figures, but why use 18th Century methods, as FDR said to Churchill?[17] Use the even divisions of great circles directly.[18]

OK, who's next? The cube and octahedron in the 4-hoop cuboctasphere are next. This is a little easier. The cube fits into the cuboctasphere with its 8 vertices in the centers of the 8 spherical triangles. The centers of its 12 edges hit the vertices of the cuboctasphere, and if you extend its 12 edges, you get 6 great circles. This is the same pattern as before, but with fewer components.

The octahedron is a different kettle of fish. It fits into the cuboctasphere all right: the 6 vertices in the center of the 6 spherical squares of the cuboctasphere, with the center points of the 12 edges at the vertices of the cuboctasphere. *But* you don't have to extend the edges to make complete great circles. They already are complete great circles, because the octahedron, in spherical form, is also the tetritetrasphere, the three-great-circle figure of the tetrahedron family. In the icosidodecasphere, you had 15 additional great circles, each shared by the icosahedron and the dodecahedron. In the cuboctasphere, you have 6 great circles used by the cube, and another 3 by the octahedron, for a total of 9. Nonetheless, the cube and octahedron are generated by the 4 great circles of the cuboctasphere with exactly the same method that created the dodecahedron and icosahedron.

For the tetritetrasphere, we almost get back to normal. If you put a tetrahedron in a tetritetrasphere, its 4 vertices go into alternating spherical triangles, and the centers of its edges map to the vertices of the tetritetrasphere. Extend the edges of the tetrahedron and you get 6 great circles. The other tetrahedron fits into the unused spherical triangles of the tetritetrasphere, and its edges lie in the same 6 great circles as the first tetrahedron's do.

This is the least-action pattern. 6 regularly divided great circles generate 15 others, which define the dodecahedron and icosahedron. Four regularly divided great circles generate 9 others, which define the cube and octahedron; and 3 regularly divided great circles generate 6 others, which define both tetrahedra. That's the pattern. The irony here is that the 6 other great circles that define the cube are the same 6 great circles that define both tetrahedra, but they in no way resemble the regularly divided arrangement of 6 great circles that are the icosidodecasphere. The cube/tetrahedral sharing of the same irregular set of 6 great circles, is why you can put two tetrahedra in a cube, as in the Moon/Hecht model of the nucleus of the atom.[19]

In the middle of all these lovely trees, I remembered something about a forest. The reason that I started investigating Archimedean solids in the first place was because the rhombic dodecahedron filled space like a cube; and no other shape in the universe, which had only a single kind of face, did that. It was as obvious as the nose on my face, that the rhombic dodecahedron isn't an Archimedean solid at all. It doesn't have regular faces. It is the dual of an Archimedean.

What About the Duals?

So, I constructed the Archimedean duals, too, all of them.[20] (See Figure 28.)

The way Archimedean dual polyhedra relate to the Archimedeans is instructive. The sphere that encloses and touches each vertex of an Archimedean solid touches the center of each face of the dual. All of the faces of a dual are the same shape, although some of them can be flipped over in a left-handed/right-handed way; and none of their faces is regular. As we will see, the Archimedean duals are harder to discuss, because of the irregularity of the faces, but I've come to believe that they are, at the very least, as important as, and as primary as, the Archimedean solids themselves.

The last dual solid I made, the disdyakis triacontahedron, was the dual of the truncated icosidodecahedron. It has 120 identical little right triangles for faces. As I was putting it together (I actually cut out 120 triangles and taped them together), I realized that the edges of this polyhedron were also great circles. That seemed interesting, but this was such a busy construction, that I couldn't see exactly what I had made at the time. (This realization also points out the importance of actually constructing the real polyhedra, rather than just looking at them.)[21]

I thought about the great-circle question for days. I had my whole set of 48 polyhedra hanging in my bedroom. There were a heck-of-a-lot of great circles dividing up the disdyakis triacontahedron into 120 triangles. Were there 15 great circles in the disdyakis triacontahedron? Were they the same 15 great circles that define the dodecahedron and the icosahedron? Could that be possible? Was the universe designed with such precision and charm that the process that created the dodecahedron and the icosahedron directly mapped to the dual of the truncated icosidodecahedron? It seemed like it should be, but was almost too much to hope for.

I went to sleep one Saturday night thinking that, if the families of polyhedra were indeed symmetrical, and the disdyakis triacontahedron was really mapped this way, then the edges of the dual of the truncated cuboctahedron, the disdyakis dodecahedron, should be made out of the 9 great circles used to make the cube and octahedron. In addition, the edges of the dual of the

Tetrakis hexahedron
(6 circles)

Disdyakis dodecahedron
(9 circles)

Disdyakis triacontahedron
(15 circles)

Figure 36
THE ARCHIMEDEAN DUALS AND THE GREAT CIRCLES
The dual of each and every Archimedean solid is directly mapped by the 15, 9, and 6 great circles derived from the 6, 4, and 3 evenly divided great circles—except for the pesky snubs.

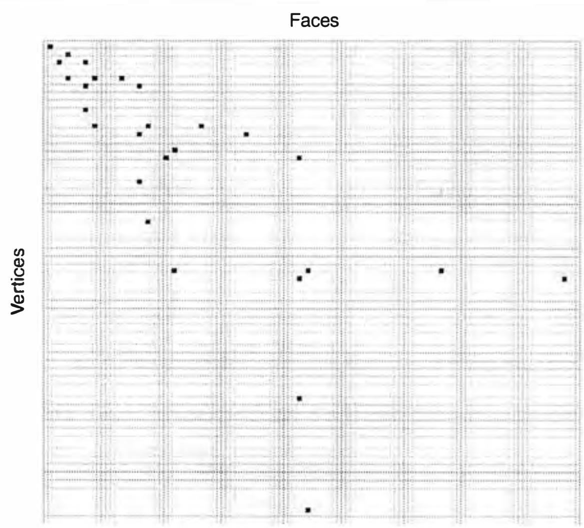


Figure 37
TWO-DIMENSIONAL GRID OF PLATONIC AND ARCHIMEDEAN POLYHEDRA AND THEIR DUALS

enharmonic truncated tetritetrahedron, the tetrakis hexahedron (which looks like a truncated octahedron), should map to the 6 great circles that make the two tetrahedra. When my eyes opened on Sunday morning, I was looking right at the tetrakis hexahedron. I saw the 6 great circles in the figure as plain as day.

This really got me moving.

It turns out that the dual of each and every Archimedean solid is directly mapped by the 15, 9, and 6 great circles derived from the 6, 4, and 3 evenly divided great circles; all of the duals except for the pesky snub cube and snub dodecahedron, are right there. The snubs are a special case, which will become more apparent, the more work we do. The duals of the truncated quasis use all of the faces defined by the great circles. The duals of the others combine some of the faces to make up rhombic and differently shaped triangular faces. It is worth the trouble of constructing the Archimedean duals just to see how this works.

The realization of the role of the great circles in the construction of the Archimedean duals made me determined to integrate the Archimedean duals into my system. How do the duals map onto the grid on which I had already placed the Archimedean solids?

These names mean something about the number of faces.

The dual of the truncated tetrahedron, the triakis tetrahedron, looks like a tetrahedron that has each face divided into three faces. There are three identical triangles in each original face of the tetrahedron. Their edges go from the center of its face to the vertex, and are pushed out a little at the center of the tetrahedron's face. I suppose the "triakis" means 3 and "tetrahedron" means, as we know, 4-sided. The first pair of these polyhedra I made were out of black poster paper, as their faces were triangles.[22] The other names are also as instructive, but not very catchy.

What you see when you add the Archimedean duals to the map, is a symmetrical pattern, like a Rorschach ink-blot test made up of dots.[23] (See Figure 37.)

As I started to do the additional mapping, I decided to go whole hog. You may remember that there are two infinite series of Archimedean solids that we have ignored so far, the prisms and anti-prisms. A prism can be constructed by taking any regular polygon from an equilateral triangle up to an equilateral bazillion-sided figure. Hang squares with edges the same length as those of your original polygon on that polygon, so each edge of the square touches an edge of the polygon and the two adjacent squares. Then put a polygon just like your original one on the bottom, and you have an Archimedean prism. (See Figure 38.)

For example, you can start with a regular hexagon, hang squares from each of its edges, and put another hexagon on the bottom. It looks like a hatbox. This is an Archimedean solid too. One sphere would touch each vertex, and one sphere would touch the center-points of both hexagons, and one sphere would touch the center-points of all squares. This is true for all of the Archimedean prisms. As you get into the higher numbers of sides, the prisms get thinner and thinner, eventually resembling a coin, or CD. As for the duals, the prism duals are all made up of isosceles triangles. The dual of the 6-sided prism, the hexagonal dipyramid, would have 12 triangles—6 isosceles triangles pointing up, like a tee-pee, and 6 pointing down. See: "dipyramid," two pyramids stuck together at their bases. As you add more edges, the triangles get longer and longer until they take on the aspect of a stretched-out dowel with sharpened ends, until you finally give up because there are too many sides.

Anti-prisms are similar to prisms, but are made with triangles rather than squares, hanging from any regular polygon—from an equilateral triangle on up. There are as many triangles as there are edges on both the top and bottom polygon. The triangles are put together alternately, so that they look like a child's drawing of shark's teeth. A 6-sided anti-prism has 12 equilateral triangles around the circumference, and hexagons on both top and bottom. The dual of an anti-prism is made up of a 4-sided figure that looks like an arrowhead. They are called trapezohedrons. The more sides the anti-prism has, the more pointy the arrowhead. The dual of the 6-sided anti-prism has 12 faces: 6 arrowheads pointing up, meeting at their points, and 6 pointing down.

There is a pattern here: the faces of the prisms are always two more than the number of polygonal edges, the vertices are always twice the number, and the edges are three times the number.

Figure 38 (b) shows the progression of the anti-prisms. The pattern here is: The faces of the anti-prisms are always two more than twice the number of the polygonal edges, the vertices are always twice, and the edges are four times the number.

To chart the duals of the prisms, switch the face and vertex numbers, just as with every other polyhedron.

There is something going on that I haven't mentioned yet: The 4-prism is the cube and the 3-anti-prism is the octahedron. Look at all the work the dual-pair of the cube and octahedron do. First, they each are Platonic solids and duals of each other. Second, the octahedron is also the tetritetrahedron, the figure directly created by the even divisions of three great circles, and parent of the tetrahedron family; and the cube is its dual, perhaps called the rhombic hexahedron in that incarnation. Third, the cube is the 4-prism, one of that infinite series; and the octahedron is its dual, a dipyramid—the one

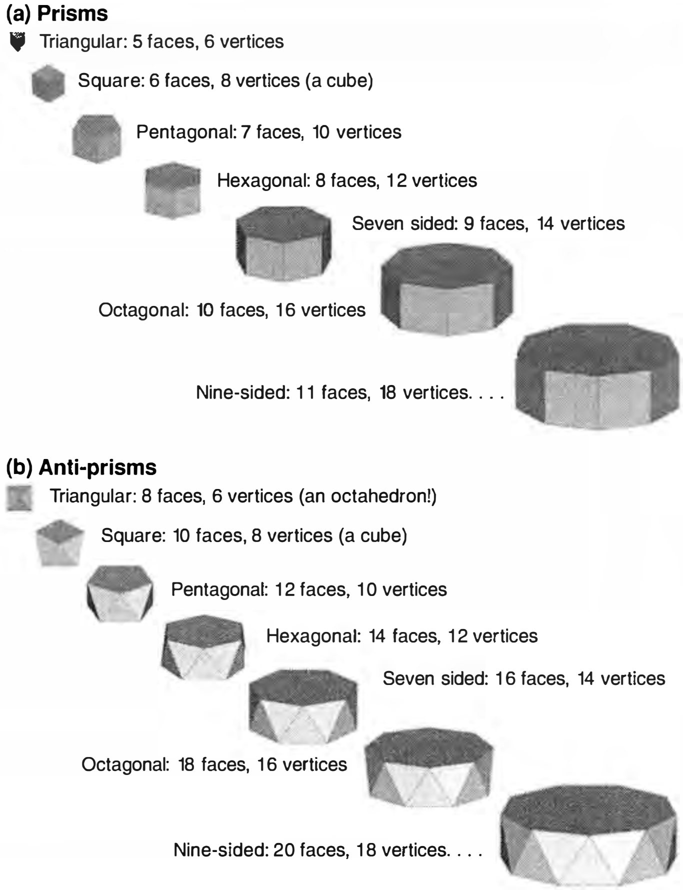


Figure 38
PRISMS AND ANTIPRISMS
The series of prisms and antiprisms goes on infinitely.

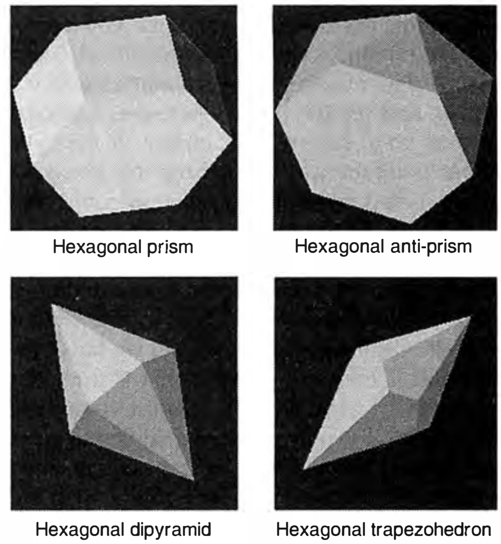


Figure 39
MORE PRISMS
Pictured here are a 6-sided prism, anti-prism, and their respective duals.

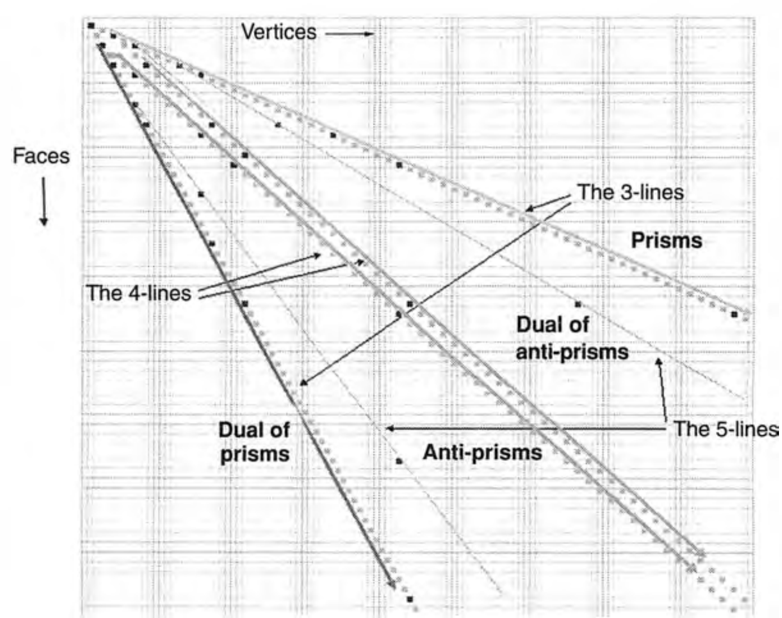
with equilateral triangular faces. Fourth, the octahedron is the three-anti-prism, the first of that infinite series; and the cube is its dual, a trapezohedron with equilateral faces.

Let's go to the grid. (See Figures 37-38.)

The Chart

There is quite a lot going on here, so I'll try to break it down. The dominant thing you see after you

Figure 40
CHARTING THE PRISMS AND THEIR DUALS
When the faces and vertices are gridded, the prisms and their duals go off in two different straight lines that seem to start at the tetrahedron. At the second prism dual-pair—the 4-prism (cube)—the anti-prisms and their duals start, with the 3-anti-prism (octahedron), which is also the dual of the 4-prism. The entire chart is contained in three pairs of straight lines. The prism and dual-of-prism lines—the "3-lines"—meet at the tetrahedron; the anti-prism and dual-of-anti-prism lines—the 4-lines"—run parallel, and very close to the "dual line," while the "5-lines" connect all three snubs and their duals, and meet the "3-lines" at the dodecahedron and icosahedron (the snub tetrahedron).



put all the dots on the graph paper, is the wedge that the prisms and their duals make. Since you are mapping dual-pairs, the chart is completely symmetrical. There is an imaginary line down the center of the pattern where you could put a mirror, and see the place where the dual of every mapped point on your side would appear in the mirror. The only mapped point that actually falls on this line is that of the tetrahedron, as it is the dual of itself. You could also call this line the pyramid line, as any pyramid you can construct would fall on this line. Pyramids are all duals of themselves. A pyramid with a million-sided base would have a million-and-one faces, and a million-and-one vertices, with 2 million edges. The tetrahedron is the simplest pyramid we have, with a base of 3 sides, and is the only pyramid that is a regular polyhedron. Since every pyramid is the dual of itself—and even though the tetrahedron is the only pyramid qualified to be mapped on our chart—they all would map right down the center dual line, if we bothered. You could fold the chart in half on the dual line, or pyramid line, and every other polyhedron would touch its dual.

The prisms and their duals go off in two different straight lines that seem to start at the tetrahedron. At the second prism dual-pair, the 4-prism (cube), the anti-prisms and their duals start with the 3-anti-prism (octahedron), which is also the dual of the 4-prism. They run in parallel lines very close to the pyramid line.

This intersection spot, where the cube and octahedron are, is the location of the most intersections of functions on this chart. Does that have something to do with the ease with which we conceptualize a cube? Cubes are easy to picture: Up, down; front, back; left, right.

The whole chart represents the boundary layer between our perceived universe, and the unseen process of creation. In dissecting this wonder, we find the snubs, and the dodecahedron family as a whole, on the far side of the singularity from us—the "dark side of the Moon," if you will. The cube, in contrast, is the nearest and most familiar point in this process. (Can singularities have sides?)

All of the Archimedean duals which have 3-sided faces occupy the same spot on the graph as a dual of an Archimedean prism, even though they are not the same shape (except the octahedron, which is the dual of a prism—the cube). All of the Archimedean polyhedra which pair with those duals fall on the same spot as one of the prisms. These are the truncated quasis and the truncated Platonics. (The truncated icosidodecahedron maps to the same location as the prism with 60-sided faces; the truncated dodecahedron and truncated icosahedron map to the prism with 30-sided faces; the truncated cuboctahedron maps to the prism with 24-sided faces; the truncated cube and truncated octahedron (truncated tetritetrahedron) map to the prism with 12-sided faces, the truncated tetrahedron maps to the prism with 6-sided faces). The duals of the Archimedeans match the duals of the prisms.

All Archimedean duals which have 4-sided faces fall on the same spot on the graph as a dual of an Archimedean anti-prism. The Archimedeans which pair with those duals co-occupy a spot with the anti-prisms themselves. These are the rhombi-quasis and the great-circle figures (the rhombicosidodecahedron maps to the anti-prism with 30-sided faces; the icosidodecahedron maps to the anti-prism with 15-sided faces; the rhombicuboctahedron maps to the anti-prism with 12-sided faces; the cuboctahedron (rhombitetritetrahedron) maps to the anti-prism with 6-sided faces; and the tetritetrahedron (octahedron) maps to the most famous prism of all, the cube).

The Archimedean duals, like all duals, owe the shapes of their faces to the nature of the vertices of their dual-pairs, and vice versa. An octahedron has faces made up of equilateral triangles, whereas the cube has 3 edges meeting at equal angles. The duals of the great-circle figures all have 4-sided faces, because the great circles meet, creating four angles. The rhombi, and all truncated Archimedean duals have 3-sided faces.

Only the snubs and their duals, which have 5-sided faces, fall on the chart in a place not already defined by the prisms or anti-prisms. Even they lie on their own straight line on the chart which intersects the prism line at the icosahedron. This implies that the snubs make up a category of their own. The whole chart is contained in three pairs of straight lines. The prism and dual-of-prism lines, the "3-lines," meet at the tetrahedron; the anti-prism and dual-of-anti-prism lines, the "4-lines," run parallel, and very close to the "dual line," and meet the "3-lines" at the cube and octahedron, while the "5-lines" connect all three snubs and their duals, and meet the "3-lines" at the dodecahedron and icosahedron (the snub tetrahedron).

The separation of the "5-lines" of the snubs is another example of their uniqueness.[24] It is not that they are snubbing the other polyhedra, of course, but there should be another infinite set of polyhedra, which would fall under the snub polyhedra. They would be like the prisms and anti-prisms, except with five-sided duals. They don't exist because they are not constructable in the discrete universe. The snub polyhedra are as close as you can come, because of the limit imposed by the nature of space. My opinion is that the angel in Dürer's *Melancolia* is trying to construct such a set, but is frustrated by the limits of physical space, and is thus, melancholy. The dual of what the angel has made in the woodcut would have 3-sided faces, at any rate, and such a series would show up on my chart at the same location as every other prism, and not on the 5-line at all. This just shows how impossible the project is.

Where the Platonic solids fall on this chart, is highly instructive, and can be understood in the context of the next paragraphs.

Once you map the Archimedeans and their duals, you can answer the question I asked about the location of the Platonics in that scheme. Do you remember when we superimposed the three families of Archimedeans? The dodecahedron, cube, and tetrahedron all fell in the same spot, but the octahedron and icosahedron seemed to randomly miss the target. The dinosaur mouths were open different amounts. Well, do the same superimposed mapping with the duals of the Archimedeans and the Platonics. The icosahedron, octahedron, and tetrahedron all map to the same place, and the dodecahedron and cube splatter somewhere else.

This is awesome.

From the perspective we have just established, the cube and dodecahedron belong to the same set of polyhedra as the Archimedean solids, while the icosahedron and octahedron belong with the Archimedean duals. If you map the Platonic polyhedra that way, the families are completely symmetrical, and once again the beauty of creation has smashed one of my

Figure 41

Melancolia, *by Albrecht Dürer. Notice the large polyhedron behind the figures.*

pet theories into the mud.

When the icosahedron and octahedron enharmonically act as Archimedean solids themselves, as snub tetrahedra and the tetritetrahedron, then they map as Archimedeans and the dodecahedron and cube map as Archimedean duals. The tetrahedron, as the point of the wedge on our graph, and dual of itself, participates in both sets.

The Platonic solids all occupy the 3-lines. The icosahedron and dodecahedron occupy the 5-lines as well, because the dodecahedron is a 5-sided-face dual of the snub tetritetrahedron (icosahedron). The cube and octahedron occupy the 4-line as well, because the cube is a 4-sided dual of the tetritetrahedron (octahedron). Most ironically, all the lines intersect at the tetrahedron, even though it is neither a prism nor the dual of a prism.

The Chart in the Back of the Book

This is a lot to keep in your head. When I was reviving my activity with the Archimedean families, a way of keeping the families and their relationships straight in my mind came to me. Don't tell anyone this trick, until they have done all the above work.

The number of edges of each member of the Archimedean families is evenly divisible by 6. If you divide each polyhedron's edge-number by 6 and look at the results as a one-dimensional graph, the tetrahedron family falls on 1, 2, 3, 4, 5, and 6. The cube family falls on 2, 4, 6, 8, 10, and 12; while the dodecahedron family falls on 5, 10, 15, 20, 25, and 30. The cube/tetrahedron enharmonic intersections are at 2, 4, and 6; with the dodecahedron/tetrahedron intersection at 5. That's it. The fact that both the cube and dodecahedron family have members with edges of 10 does not indicate an enharmonic intersection; they just have the same number of edges.

The utilization of the edge-axis in this way is why, when I first started mapping the Archimedean families, it was most convenient to use the faces and vertices for a two-dimensional view. The polyhedra seemed to bunch up in the edge-axis view, and made the chart sloppy. I thought that was a problem, and went on to do all the work recounted above. If I had realized that only using the edges for mapping, I could show both the symmetry and intersections of the families, I would have missed all this fun.

You can discourse on this topic, off the top of your head with this simple chart in your mind.[25] Or draw it out: 1, 2, 3, 4, 5, and 6 down the center of a piece of paper; 2, 4, 6, 8, 10, 12 on the right side; and 5, 10, 15, 20, 25, 30 on the left. Make sure that the numbers are lined up, 2 next to 2, 4 next to 4, and so on; circle all 2's, 4's, 5's, and 6's, and you're done. See Figure 40.

Once the idea is in your head, this is the only mnemonic device you will need.

So, What Do We Have?

In summary, we have created two sets of tools, useful in the philosophical examination of geometry, and, I might add, just as useful in the geometrical examination of philosophy.

The first set is the collection of great-circle figures: 3, 4, and 6 even divisions of great circles by other great circles, from which we create the 6, 9, and 15 other great-circle arrangements which give you the Archimedean duals, and the Archimedean polyhedra arranged in the three symmetrical families. The great circles are useful in the planning and construction of our polyhedra. All of these collections of great circles together, I've come to call "God's graph paper." (Figure 35).

The other set of tools is the mapping of the locations of the polyhedra onto a three-dimensional grid. You have the three families of Platonic and Archimedean solids, which look like three constellations, and show the symmetry and intersections of the families. Adding the duals of the Archimedean solids shows how the dual-pairs are mirror images of each other, while adding the prisms, anti-prisms, and their duals provides a framework for the other polyhedra, and highlights some of the processes that create the shapes. The various stages of this mapping are useful in seeing what has been constructed.

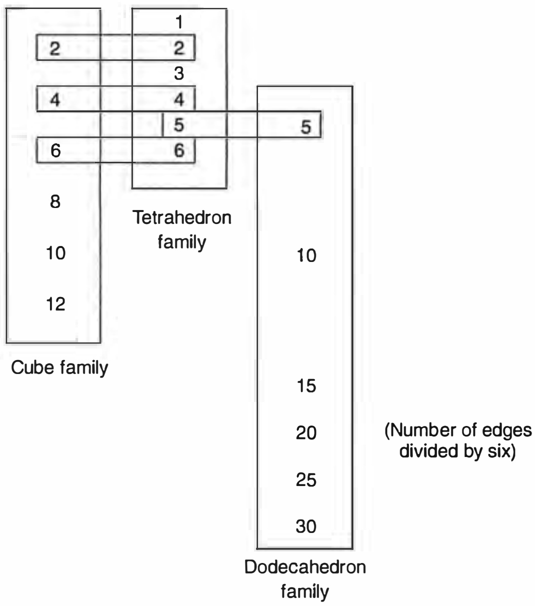


Figure 42
CHARTING THE FAMILIES OF POLYHEDRA
This is the chart at the back of the book, showing how the families of polyhedra intersect. The number of edges of each member of the Archimedean families is evenly divided by 6. If you divide each polyhedron's edge-number by 6 and plot the results in one dimension, this is the result.

Gridding or mapping the positions of the polyhedra is a tool to examine the limits embedded in visible space. Don't look at the graph as a thing. It is picture of a small part of the ongoing process of creating the universe. Your examination of the chart is part of that process of creation. It would be nice to build a chart big enough to put models of the polyhedra where they appear on the grid. Even if we do that, even if we have a few city blocks to landscape, and the chart is big enough to walk around in, it won't be a *thing*. Imagine walking along the 3-line by each of the prisms, past the dodecahedron and the truncated tetrahedron, until you reach the cube. You stop and look across the little stream that represents the pyramid line, seeing the octahedron and the anti-prism row leading off to your right, and reflect on how many things the octahedron is doing at the same time, even while it appears to be just sitting there: Your thoughts at that moment are what's happening, not the models themselves.

These are really tools you can use to answer questions such as, how is the axis of symmetry different in the dodecahedron vs. the rhombic dodecahedron? They both have 12 faces, which are different shapes. How could there possibly be two dodecahedra with differently shaped faces? The Composer didn't sit down and cut out cardboard. How do the faces orient to each other in each polyhedron? Look at the 3-hoop and 9-hoop spheres. Clearly, the center of each face of the rhombic dodecahedron falls at the center of each edge of the tetritetrasphere, the evenly divided 3-hoop construction. Now look at the dodecasphere in the 15-hoop sphere. The center of each face of the dodecahedron also falls on an edge of the 3-hoop tetritetrasphere, but not in the center of the arc segment. Could it be that the center of the face divides the edge at the Golden Mean? I think it does. When you divide the arcs thusly, you have to choose either a right-handed or left-handed orientation. This is another indication of the dodecahedron family's affinity to the snub figures. Try picturing that without the great-circle constructions as a guide.

The relationships presented here are true, but what is the relevance? How that works is up to you. The last thing you want is a well-stocked tool box sitting unused in a closet. Make, or borrow an hypothesis and then do the constructions. Once you get the ball rolling, it becomes a self-feeding process.

As a final inspiration, some wisdom from Act I, Scene 5 of Mozart's opera *Don Giovanni*. Don Giovanni (Don Juan) foolishly lets himself get within arm's reach of a former, abandoned lover who is looking for him to make him marry her. He wants to have his servant, Leporello, save him by distracting her by recounting his lengthy list of Giovanni's amorous adventures:

He says (loosely), "Tell her everything."

Leporello, missing the point, either on purpose, or not, asks, "Everything?"

"Yes, yes, tell her everything."

"And make it snappy," she interjects.

"Well, ma'am, in this world, truly," says the embarrassed Leporello, "a square is not round."

See, everybody used to know that geometry was the beginning of "everything."[26]

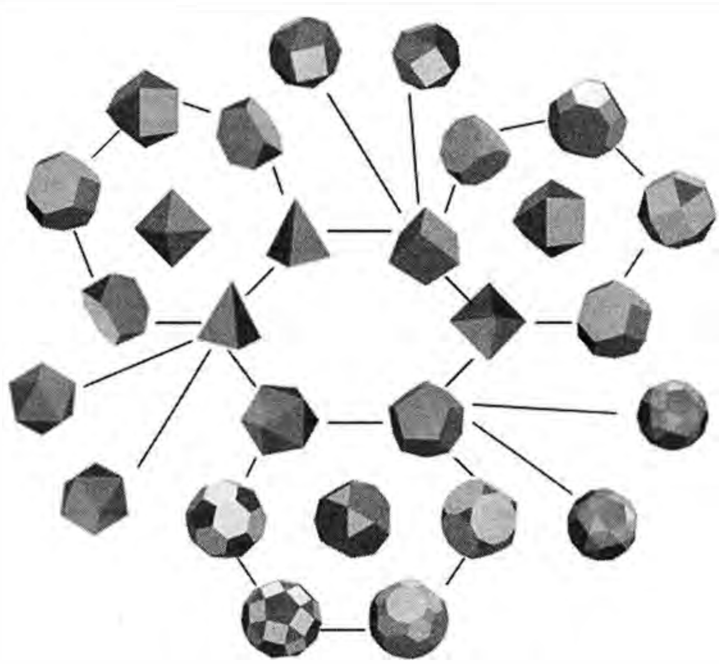


Figure 43
THE SHAPE OF SPACE II
The beginning of everything!

Notes

1. Max Casper (1880-1956), *Kepler,* (Mineola, N.Y.: Dover Publications, Inc. 1993), p. 380.
2. Edwin A. Abbot (1838-1926), *Flatland, A Romance of Many Dimensions, with Illustrations by the Author, A. Square,* http://www.geom.uiuc.edu/›banchoff/Flatland/.
3. Lyndon H. LaRouche, Jr., "On the Subject of Metaphor," *Fidelio,* Vol. 1, No. 3, Fall 1992.
4. Johannes Kepler (1571-1630), *The Six-cornered Snowflake,* edited by Colin Hardie, (New York: Oxford University Press, 1966), 74 pp. This is a beautiful work.
5. My friend, Jacob Welsh, (b. 1989) has correctly pointed out that if you squash a cube the right way, and make all the faces identically diamond-shaped, this figure will also fill space. I maintain that a squished cube is still a cube, so let the merriment continue.
6. Magnus J Wenninger, *Polyhedron Models,* (New York: Cambridge University Press, 1971), pp. 12-13. Wenninger gives brief notes on construction techniques. My current practice follows his closely; however, when I started this project, I used wide, clear tape to cover each cardboard face, and then taped the individual faces together.
7. Plato, *Parmenides, or On Ideas, Logical,* unpublished translation from 1990 by Leslie B. Vaughan.
8. To show what I mean by this, here are some news reports from the internet:

   "One Critically Wounded during Rotterdam Soccer Riot—Rioting soccer fans may have returned fire at police during a clash in central Rotterdam Sunday night in which police shot into the crowd, wounding four people. . . ."

   "Rome Soccer Riot Was Planned—A hardcore of Lazio and AS Roma soccer fans worked together to spark the riot that caused the Rome derby to be abandoned on Sunday, politicians said on Monday. Police on Monday said they had arrested 13 supporters from both sides, some of them known hooligans, following a 6-hour pitched battle between police and fans that left more than 170 people injured. . . ."

   "Soccer Riot in Russia Kills One—Russian soccer fans rampaged near the Kremlin after their team lost to Japan in the World Cup on Sunday, setting fire to cars, smashing store windows and attacking a group of young Japanese musicians. At least one man was killed in the melee. . . ."
9. The late, great Fred Wills (1928-1992), former Foreign Affairs and Justice Minister of Guyana, hero of the Colombo conference and the Non-Aligned Movement, Shakespearean scholar, friend of Lyndon LaRouche, and cricket expert of international renown, taught me that the above quote is almost always useful, and incidentally, the oldest rhetorician's trick in the book. If someone has the misfortune to speak before you speak, you are in the perfect position to trump his or her lead, like Regan does to the other "bad sister," Goneral, in *King Lear.* You can adopt all their hard work as your own. Anything you add is by definition more than your unfortunate predecessor has said.

   That is my role here, to build on the work that our "non-checkerboard" faction has already done over the last 3,000 years or so. With any luck at all we can end up in better shape than Regan does later in *King Lear.* ("Sick, oh sick.")
10. Three symmetrical families; three pairs of Platonic solids; three sets of great circle figures: What is with all these threes? Now look, when we say that the cube has 6 faces, we don't mean "6" in a *Sesame Street,* "one, two, three, four, five, s-i-i-i-x," kind of way. The 6 faces of a cube are oriented a certain way: up, down, left, right, front, back. This 6 is not just a dead number, but is in the process of going from somewhere to somewhere. In fact, you wouldn't have those concepts of direction without a reference like the cube. If fact, the structure of the universe, which we are investigating here, determines where those 6 faces fall, and how they are shaped, and why they are unique. 6 doesn't just mean 6, at all. Likewise, the three families of Archimedean polyhedra are not just three, as in "three." The past is not the future, and certainly bears no topographical resemblance to the present. Past, present, future, your whole existence is shaped by the idea of three, but it is not just "a three."
11. *Theropods* (meaning "beast-footed") were a sub-order of Saurischian dinosaurs. They were fast-moving, bipedal carnivores (meat-eaters) with grasping hands and clawed digits. They looked like the kind of turkeys that could have *you* for Thanksgiving.
12. *Deinonychus antirrhopus,* "Terrible Claw upturned" was supposedly a lightly built, fast-moving, agile, bipedal (walked on two legs), bird-like dinosaur, which could grow up to 10 feet long and lived from 110 to 100 million years ago.
13. *Ceratosaurus,* the "Horn Lizard" is said to have been a powerful predator that walked on two strong legs, had a strong, "s"-shaped neck, and had a short horn on its snout. The *Ceratosaurus* lived from 156 to 145 million years ago and could be 15-20 feet long.
14. *Tyrannosaurus rex,* the "Tyrant lizard king," was a huge meat-eating dinosaur that lived during the late Cretaceous period, about 85 million to 65 million years ago. Until recently, *Tyrannosaurus* rex was the biggest known carnivorous dinosaur, at 40 feet long. Current teaching has it that the *Giganotosaurus* and *Carcharodontosaurus* are slightly bigger.
15. Friedrich Schiller, "What Is, and to What End Do We Study Universal History? 1789 Inaugural Address at Jena," translated by Caroline Stephan and Robert Trout, *Friedrich Schiller, Poet of Freedom, Vol. II,* (Washington, D.C.: Schiller Institute, 1988), pp.254-255.

    The course of studies which the scholar who feeds on bread alone sets himself, is very different from that of the philosophical mind. The former, who for all his diligence, is interested merely in fulfilling the conditions under which he can perform a vocation and enjoy its advantages, who activates the powers of his mind only thereby to improve his material conditions and to satisfy a narrow-minded thirst for fame, such a person has no concern upon entering his academic career, more important than distinguishing most carefully those sciences which he calls "studies for bread," from all the rest, which delight the mind for their own sake. Such a scholar believes, that all the time he devoted to these latter, he would have to divert from his future vocation, and this thievery he could never forgive himself.
16. Peter Cook (1937-1995), "Sitting on the Bench," *Beyond the Fringe,* (New York: Samuel French, Inc. 1963).

    Strange, but not odd, that a Cambridge-educated comedian would use this as a joke title for a book in a comedy review.

    ". . .I am very interested in the Universe and all that surrounds it. In fact, I'm studying Nesbitt's book, *The Universe and All That Surrounds It.* He tackles the subject boldly, goes from the beginning of time right through to the present day, which according to Nesbitt is Oct. 31, 1940. And he says the Earth is spinning into the Sun, and we will all be burnt to death. But he ends the book on a note of hope, he says, 'I hope this will not happen.' "
17. Elliott Roosevelt, *As He Saw It, The Story of the World Conferences of FDR,* (New York: Duell Sloan and Pearce, 1946), p. 36.
18. This is what LaRouche says to do in the "Metaphor" paper, but this is not how he says to do it. He says, "From the 6-hooped figure containing dodecahedron and icosahedron, the cube, octahedron, and tetrahedron may be readily derived." And it can. *However,* you may see how the Platonic and other polyhedra may be formed from the three sets of evenly divided great circles.
19. See Laurence Hecht and Charles B. Stevens, "New Explorations with the Moon Model," *21st Century,* Fall 2004.
20. Magnus J. Wenninger, *Dual Models,* (New York: Cambridge University Press, 1983), pp.1-6. Wenninger gives two methods to determine what the dual of any polyhedron is. Going through this with a group of people would make an interesting class.
21. Robert Williams, *The Geometrical Foundation of Natural Structure,* (Mineola, N.Y.: Dover Publications, Inc. 1972), pp.63-97. This section of Mr. Williams's book was significantly valuable to me when I first started constructing polyhedra. In particular, the face-angles of the dual polyhedra made this portion of the project possible, before I had read Wenninger's *Dual Models* book referenced in footnote 19.
22. Don't make a polyhedron all black, unless you are going to hang it in a nightclub. I was trying to highlight the fact that the Archimedean duals are made up of only 3-, 4-, or 5-sided faces by making them black, gray, or white, depending on how many sides the faces of the polyhedron had. However, you can't see what the black ones look like in a photograph. They do look mighty slick in person, though.
23. When I first saw the pattern, I thought it looked like a sampling of an amplitude-modulated envelope of increasing amplitude, running for three-and-a-half cycles of the modulating frequency. I later imagined that each family of Archimedean solids and their duals could be connected by a pair of sine waves 180 degrees out of phase with each other, either expanding from, or contracting on, the Platonics, for three or four cycles. I am far from complete in connecting each family's dots with curves, or sine waves, rather than dinosaur skeletons. It is more of an artistic proposition, than a scientific one. That could be because I haven't seen the pattern correctly. Perhaps a bright young person with a fancy computer program, or even a bright old person with a slide rule, could tidy this up.
24. My friend Gerry Therrien has spoken of how Kepler wrote about the attributes and genesis of the snub polyhedra. I hope he writes up his observations sometime. For now, look at the snub figures and then at any anti-prism, and ponder the similarities.
25. Plato, *Meno.*

    This is funny: Plato has Meno express amazement that Socrates can't even tell him what virtue is, as Meno has spoken "at great length, and in front of many people on the topic." Later, when Socrates shows the slave why doubling the sides of a square won't double the area, Socrates says that, just a moment before the slave would have spoken at great length, and in front of many people on doubling the side of a square. Yes, Socrates did irritate a few people.
26. DON GIOVANNI: Si, si, dille pur tutto.

    (Parte non visto da Donn' Elvira.) DONNA ELVIRA: Ebben, fa presto. LEPORELLO: (Balbettando): Madama. . . veramente. . . in questo mondo conciossiacosaquandofosseché. . . il quadro non è tondo. . . .